2021

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AP® Calculus AB

Sample Student Responses and Scoring Commentary

Inside:

Free Response Question 1

☑ Scoring Guideline

☑ Student Samples

☑ Scoring Commentary

Part A (AB or BC): Graphing calculator required
Question 1 — 9 points

General Scoring Notes

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

Scoring guidelines and notes contain examples of the most common approaches seen in student responses. These guidelines can be applied to alternate approaches to ensure that these alternate approaches are scored appropriately.

r (centimeters)	0	1	2	2.5	4
$f(r)$ (milligrams per square centimeter)	1	2	6	10	18

The density of a bacteria population in a circular petri dish at a distance r centimeters from the center of the dish is given by an increasing, differentiable function f, where $f(r)$ is measured in milligrams per square centimeter. Values of $f(r)$ for selected values of r are given in the table above.

Model Solution	Scoring	
(a) Use the data in the table to estimate $f'(2.25)$. Using correct units, interpret the meaning of your answer in the context of this problem.		
$f'(2.25) \approx \dfrac{f(2.5)-f(2)}{2.5-2} = \dfrac{10-6}{0.5} = 8$	Estimate	**1 point**
At a distance of $r = 2.25$ centimeters from the center of the petri dish, the density of the bacteria population is increasing at a rate of 8 milligrams per square centimeter per centimeter.	Interpretation with units	**1 point**

Scoring notes:

- To earn the first point the response must provide both a difference and a quotient and must explicitly use values of f from the table.
- Simplification of the numerical value is not required to earn the first point. If the numerical value is simplified, it must be correct.
- The interpretation requires all of the following: distance $r = 2.25$, density of bacteria (population) is increasing or changing, at a rate of 8, and units of milligrams per square centimeter per centimeter.
- The second point (interpretation) cannot be earned without a nonzero presented value for $f'(2.25)$.
- To earn the second point the interpretation must be consistent with the presented nonzero value for $f'(2.25)$. In particular, if the presented value for $f'(2.25)$ is negative, the interpretation must include "decreasing at a rate of $|f'(2.25)|$" or "changing at a rate of $f'(2.25)$." The second point cannot be earned for an incorrect statement such as "the bacteria density is decreasing at a rate of -8 … " even for a presented $f'(2.25) = -8$.
- The units ($\text{mg/cm}^2\text{/cm}$) may be attached to the estimate of $f'(2.25)$ and, if so, do not need to be repeated in the interpretation.
- If units attached to the estimate do not agree with units in the interpretation, read the units in the interpretation.

Total for part (a) 2 points

(b) The total mass, in milligrams, of bacteria in the petri dish is given by the integral expression $2\pi\int_0^4 r\,f(r)\,dr$. Approximate the value of $2\pi\int_0^4 r\,f(r)\,dr$ using a right Riemann sum with the four subintervals indicated by the data in the table.

$2\pi\int_0^4 r\,f(r)\,dr \approx 2\pi(1\cdot f(1)\cdot(1-0) + 2\cdot f(2)\cdot(2-1) + 2.5\cdot f(2.5)\cdot(2.5-2) + 4\cdot f(4)\cdot(4-2.5))$	Right Riemann sum setup	**1 point**
$= 2\pi(1\cdot 2\cdot 1 + 2\cdot 6\cdot 1 + 2.5\cdot 10\cdot 0.5 + 4\cdot 18\cdot 1.5)$ $= 269\pi = 845.088$	Approximation	**1 point**

Scoring notes:

- The presence or absence of 2π has no bearing on earning the first point.
- The first point is earned for a sum of four products with at most one error in any single value among the four products. Multiplication by 1 in any term does not need to be shown, but all other products must be explicitly shown.
- A response of $1\cdot f(1)\cdot(1-0) + 2\cdot f(2)\cdot(2-1) + 2.5\cdot f(2.5)\cdot(2.5-2) + 4\cdot f(4)\cdot(4-2.5)$ earns the first point but not the second.
- A response with any error in the Riemann sum is not eligible for the second point.
- A response that provides a completely correct left Riemann sum for $2\pi\int_0^4 r\,f(r)\,dr$ and approximation (91π) earns one of the two points. A response that has any error in a left Riemann sum or evaluation for $2\pi\int_0^4 r\,f(r)\,dr$ earns no points.
- A response that provides a completely correct right Riemann sum for $2\pi\int_0^4 f(r)\,dr$ and approximation (80π) earns one of the two points. A response that has any error in a right Riemann sum or evaluation for $2\pi\int_0^4 f(r)\,dr$ earns no points.
- Simplification of the numerical value is not required to earn the second point. If a numerical value is given, it must be correct to three decimal places.

Total for part (b) 2 points

(c) Is the approximation found in part (b) an overestimate or underestimate of the total mass of bacteria in the petri dish? Explain your reasoning.

$\frac{d}{dr}(r\,f(r)) = f(r) + r\,f'(r)$	Product rule expression for $\frac{d}{dr}(r\,f(r))$	**1 point**
Because f is nonnegative and increasing, $\frac{d}{dr}(r\,f(r)) > 0$ on the interval $0 \le r \le 4$. Thus, the integrand $r\,f(r)$ is strictly increasing. Therefore, the right Riemann sum approximation of $2\pi\int_0^4 r\,f(r)\,dr$ is an overestimate.	Answer with explanation	**1 point**

Scoring notes:

- To earn the second point a response must explain that $r\,f(r)$ is increasing and, therefore, the right Riemann sum is an overestimate. The second point can be earned without having earned the first point.
- A response that attempts to explain based on a left Riemann sum for $2\pi\int_0^4 r\,f(r)\,dr$ from part (b) earns no points.
- A response that attempts to explain based on a right Riemann sum for $2\pi\int_0^4 f(r)\,dr$ from part (b) earns no points.

Total for part (c) 2 points

(d) The density of bacteria in the petri dish, for $1 \le r \le 4$, is modeled by the function g defined by $g(r) = 2 - 16\left(\cos\left(1.57\sqrt{r}\right)\right)^3$. For what value of k, $1 < k < 4$, is $g(k)$ equal to the average value of $g(r)$ on the interval $1 \le r \le 4$?

Average value $= g_{\text{avg}} = \frac{1}{4-1}\int_1^4 g(r)\,dr$	Definite integral	**1 point**
$\frac{1}{4-1}\int_1^4 g(r)\,dr = 9.875795$	Average value	**1 point**
$g(k) = g_{\text{avg}} \Rightarrow k = 2.497$	Answer	**1 point**

Scoring notes:

- The first point is earned for a definite integral, with or without $\frac{1}{4-1}$ or $\frac{1}{3}$.
- A response that presents a definite integral with incorrect limits but a correct integrand earns the first point.
- Presentation of the numerical value 9.875795 is not required to earn the second point. This point can be earned by the average value setup: $\frac{1}{3}\int_1^4 g(r)\,dr$.
- Once earned for the average value setup, the second point cannot be lost. Subsequent errors will result in not earning the third point.
- The third point is earned only for the value $k = 2.497$.
- The third point cannot be earned without the second.
- Special case: A response that does not provide the average value setup but presents an average value of -13.955 is using degree mode on their calculator. This response would not earn the second point but could earn the third point for an answer of $k = 2.5$ (or 2.499).

Total for part (d) 3 points

Total for question 1 9 points

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Answer QUESTION 1 parts (a) and (b) on this page.

r (centimeters)	0	1	2	2.5	4
$f(r)$ (milligrams per square centimeter)	1	2	6	10	18

Response for question 1(a)

$$f'(2.25) \approx \frac{f(2.5)-f(2)}{2.5-2} = \frac{10-6}{0.5} = \frac{4}{0.5} = 8$$ milligrams per square centimeter per centimeter

The density of bacteria changes at a rate of approximately 8 milligrams per square centimeter per centimeter at distance $r = 2.25$ centimeters from the center of the dish

Response for question 1(b)

$$2\pi\int_0^4 r \cdot f(r)\,dr \approx 2\pi(2\cdot1\cdot1 + 6\cdot1\cdot2 + 10\cdot0.5\cdot2.5 + 18\cdot1.5\cdot4)$$

$$= 2\pi(2+12+12.5+108) = 2\pi(134.5) = 269\pi$$ milligrams

Answer QUESTION 1 parts (c) and (d) on this page.

Response for question 1(c)

As a rule, Right riemann Sums are always an over estimate for functions with positive slope, and underestimates for functions with negative slope. The slope of $r \cdot f(r)$ is equal to $r' \cdot f(r) + r \cdot f'(r)$. Since r, r', $f(r)$, and $f'(r)$ are always positive on the interval $[0,4]$, $r \cdot f(r)$ always has a positive slope on that interval. Since it's a positive sloped function, the right riemann sum for $r \cdot f(r)$ from 0 to 4 is an over estimate.

Response for question 1(d)

Avg of $g(r) = \frac{1}{4-1}\int_1^4 (2-16(\cos(1.57\sqrt{r}))^3)\,dr = \frac{29.627}{3} = 9.876$

$g(k) = 2-16(\cos(1.57\sqrt{k}))^3 = 9.876$

$k = 2.497$

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Answer QUESTION 1 parts (a) and (b) on this page.

r (centimeters)	0	1	2	2.5	4
$f(r)$ (milligrams per square centimeter)	1	2	6	10	18

Response for question 1(a)

$$f'(2.25) \approx \frac{f(2.5)-f(2)}{2.5-2} = \frac{10-6}{0.5} = 8 \ \frac{\text{milligrams}}{\text{cm}^3}$$

The rate of change of f(r) at distance r = 2.25 centimeters is approximately 8 milligrams per cubic centimeter

Response for question 1(b)

$$2\pi \int_0^4 r f(r)\,dr \approx 2\pi[1(1)f(1) + 1(2)f(2) + 0.5(2.5)f(2.5) + 1.5(4)f(4)]$$

$$\approx 2\pi[2 + 12 + 12.5 + 108] = 269\pi$$

Answer QUESTION 1 parts (c) and (d) on this page.

Response for question 1(c)

for [0, 4], f(r) is increasing (as is r)
so r f(r) is increasing.
As a right Riemann sum was used to take the intergal of an increasing function, it was an overestimate

Response for question 1(d)

Avg. value of g(r) on [1, 4] $= \frac{1}{4-1}\int_1^4 (2-16(\cos(1.57\sqrt{r}))^3)\,dr = 9.87579487$

$2-16(\cos(1.57\sqrt{K}))^3 = 9.875794868$

at K = 2.497 centimeters

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Answer QUESTION 1 parts (a) and (b) on this page.

r (centimeters)	0	1	2	2.5	4
$f(r)$ (milligrams per square centimeter)	1	2	6	10	18

Response for question 1(a)

$f'(2.25) = \frac{f(2.5)-f(2)}{2.5-2} \rightarrow \frac{10-6}{2.5-2} = \frac{4}{0.5} = 8$

$f'(2.25) = 8$ mg per cm² per cm²

At $r = 2.25$, the rate of the density of the bacteria population is increasing at a rate of 8 mg per cm² per cm²

Response for question 1(b)

$2\pi \cdot [1(2)+1(6)+0.5(10)+1.5(18)]$

$2\pi \cdot (2+6+5+27)$

$2\pi(40) = 80\pi = 251.327$

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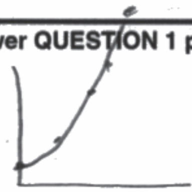
Answer QUESTION 1 parts (c) and (d) on this page.

Response for question 1(c)



The right Riemann sum approximation is an overestimate due to the fact that the total mass of the bacteria is increasing since its represented by f(r) which is the function of the bacteria's density which is an increasing differentiable function

Response for question 1(d)

$g(r) = 2 - 16(\cos(1.57\sqrt{r}))^3$

$\frac{1}{4-1}\int_1^4 g(r)\,dr = 4.875794868$

$g'(k) = 4.875794868$

Question 1

Note: Student samples are quoted verbatim and may contain spelling and grammatical errors.

Overview

The context of this problem is bacteria in a circular petri dish. The increasing, differentiable function f gives the density of the bacteria population (in milligrams per square centimeter) at a distance r centimeters from the center of the dish. Selected values of $f(r)$ are provided in a table.

In part (a) students were asked to use the table to estimate $f'(2.25)$ and interpret the meaning of this value in context, using correct units. A correct response should estimate the derivative value using a difference quotient, drawing from the data in the table that most tightly bounds $r = 2.25.$ The interpretation should explain that when $r = 2.25$ centimeters from the center of the petri dish, the density of the bacteria population is increasing at a rate of roughly 8 milligrams per square centimeter per centimeter.

In part (b) students were told that $2\pi\int_0^4 r\,f(r)\,dr$ gives the total mass, in milligrams, of the bacteria in the petri dish. They were asked to estimate the value of this integral using a right Riemann sum with the values given in a table. A correct response should multiply the sum of the four products $r_i \cdot f(r_i) \cdot \Delta r_i$ drawn from the table by $2\pi.$

In part (c) students were asked to explain whether the right Riemann sum approximation found in part (b) was an overestimate or an underestimate of the total mass of bacteria. A correct response should determine the derivative of $r \cdot f(r)$ using the product rule, use the given information that f is nonnegative to conclude that this derivative is positive and, therefore, that the integrand is strictly increasing on the interval $0 \le r \le 4.$ This means that the right Riemann sum approximation is an overestimate.

In part (d) another function, $g(r) = 2 - 16\left(\cos\left(1.57\sqrt{r}\right)\right)^3$, was introduced as a function that models the density of the bacteria in the petri dish for $1 \le r \le 4.$ Students were asked to find the value of k such that $g(k)$ is equal to the average value of $g(r)$ on the interval $1 \le r \le 4.$ A correct response should set up the average value of $g(r)$ as $\frac{1}{3}\int_1^4 g(r)\,dr,$ then use a graphing calculator to solve for k when setting $g(k)$ equal to this average value.

Sample: 1A
Score: 9

The response earned 9 points: 2 points in part (a), 2 points in part (b), 2 points in part (c), and 3 points in part (d). In part (a) the difference quotient of $\frac{10-6}{0.5}$ in the first line would have earned the first point with no simplification. In this case, correct simplification to 8 in the first line earned the first point. The response earned the second point for an interpretation of the density of the bacteria changing at 8 milligrams per square centimeter per centimeter at $r = 2.25.$ In part (b) the response earned the first point for the sum of products expression $2\pi \cdot (2 \cdot 1 \cdot 1 + 6 \cdot 1 \cdot 2 + 10 \cdot 0.5 \cdot 2.5 + 18 \cdot 1.5 \cdot 4)$ in the first line on the right. This sum of products expression would also have earned the second point with no simplification. In this case, correct simplification to 269π in the second line earned the second point. In part (c) the response earned the first point for the product rule expression of $r' \cdot f(r) + r \cdot f'(r)$ for $\frac{d}{dr}(r\,f(r))$ in the fourth line. The response earned the second point for the conclusion that $r\,f(r)$ has a positive slope because $r, r', f(r),$ and $f'(r)$ are positive on the interval and, therefore, the estimate is an overestimate. In part (d) the response earned the first and second points for the definite integral $\frac{1}{4-1}\int_1^4\left(2 - 16\left(\cos\left(1.57\sqrt{r}\right)\right)^3\right)dr$ giving the average value in the first line. The response earned the third point for the correct value of $k = 2.497$ in the third line.

Question 1 (continued)

Sample: 1B
Score: 7

The response earned 7 points: 1 point in part (a), 2 points in part (b), 1 point in part (c), and 3 points in part (d). In part (a) the difference quotient of $\frac{10-6}{0.5}$ in the first line would have earned the first point with no simplification. In this case, correct simplification to 8 in the first line earned the first point. The response did not earn the second point because the density of the bacteria is not referenced. In part (b) the response earned the first point for the sum of products expression $2\pi[1(1)f(1) + 1(2)f(2) + 0.5(2.5)f(2.5) + 1.5(4)f(4)]$ in the first line. The sum of products expression $2\pi[2 + 12 + 12.5 + 108]$ in the second line would have earned the second point with no simplification. In this case, simplification to 269π in the second line earned the second point. In part (c) the response did not earn the first point because there is no product rule expression for $\frac{d}{dr}(rf(r))$. The response earned the second point for the claim that $rf(r)$ is increasing in the second line and, therefore, the right Riemann sum is an overestimate in the fifth line. In part (d) the response earned the first and second points for the definite integral $\frac{1}{4-1}\int_1^4 \left(2 - 16\left(\cos\left(1.57\sqrt{r}\right)\right)^3\right) dr$ giving the average value in the first line. The response earned the third point for the correct value of $k = 2.497$ in the third line.

Sample: 1C
Score: 4

The response earned 4 points: 1 point in part (a), 1 point in part (b), no points in part (c), and 2 points in part (d). In part (a) the difference quotient of $\frac{10-6}{2.5-2}$ in the first line would have earned the first point with no simplification. In this case, correct simplification to 8 in the first line earned the first point. The response did not earn the second point because the response states the rate of the density of the bacteria population is increasing at a rate and because the units in the interpretation are incorrect. In part (b) the response earned one of the two points for providing a completely correct right Riemann sum for $2\pi\int_0^4 f(r)\,dr$. The sum of products expression $2\pi \cdot [1(2) + 1(6) + 0.5(10) + 1.5(18)]$ in the first line would have earned one of the two points with no simplification. In this case, correct simplification to 251.327 in the third line earned one of the two points. In part (c) the response did not earn either point because the response attempted to explain a right Riemann sum for $2\pi\int_0^4 f(r)\,dr$. In part (d) the response earned the first and second points for the definite integral $\frac{1}{4-1}\int_1^4 g(r)\,dr$ giving the average value in the second line. The response did not earn the third point because no value is given for k.

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Sample Student Responses and Scoring Commentary

Inside:

Free Response Question 2

☑ Scoring Guideline

☑ Student Samples

☑ Scoring Commentary

Part A (AB): Graphing calculator required
Question 2 — 9 points

General Scoring Notes
Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

Scoring guidelines and notes contain examples of the most common approaches seen in student responses. These guidelines can be applied to alternate approaches to ensure that these alternate approaches are scored appropriately.

A particle, P, is moving along the x-axis. The velocity of particle P at time t is given by $v_P(t) = \sin\left(t^{1.5}\right)$ for $0 \le t \le \pi$. At time $t = 0$, particle P is at position $x = 5$.

A second particle, Q, also moves along the x-axis. The velocity of particle Q at time t is given by $v_Q(t) = (t - 1.8) \cdot 1.25^t$ for $0 \le t \le \pi$. At time $t = 0$, particle Q is at position $x = 10$.

Model Solution	Scoring	
(a) Find the positions of particles P and Q at time $t = 1$.		
$x_P(1) = 5 + \int_0^1 v_P(t)\,dt = 5.370660$ At time $t = 1$, the position of particle P is $x = 5.371$ (or 5.370).	One definite integral	**1 point**
	One position	**1 point**
$x_Q(1) = 10 + \int_0^1 v_Q(t)\,dt = 8.564355$ At time $t = 1$, the position of particle Q is $x = 8.564$.	The other position	**1 point**

Scoring notes:

- The first point is earned for the explicit presentation of at least one definite integral, either $\int_0^1 v_P(t)\,dt$ or $\int_0^1 v_Q(t)\,dt$.
- The first point must be earned to be eligible for the second and third points.
- The second point is earned for adding the initial condition to at least one of the definite integrals and finding the correct position.
- Writing $\int_0^1 v_P(t) + 5 = 5.370660$ does not earn a position point, because the missing dt makes this statement unclear or false. However, $5 + \int_0^1 v_P(t) = 5.370660$ does earn the position point because it is not ambiguous. Similarly, for the position of Q.
- Read unlabeled answers presented left to right, or top to bottom, as $x_P(1)$ and $x_Q(1)$, respectively.
- Special case 1: A response of $x_P(1) = 5 + \int_0^a v_P(t)\,dt = 5.370660$ AND $x_Q(1) = 10 + \int_0^a v_Q(t)\,dt = 8.564355$ for $a \neq 1$ earns one point.
- Special case 2: A response of $x_P(1) = 5 + \int v_P(t)\,dt = 5.370660$ AND $x_Q(1) = 10 + \int v_Q(t)\,dt = 8.564355$ or the equivalent, never providing the definite integrals, earns one point.
- Degree mode: A response that presents answers obtained by using a calculator in degree mode does not earn the first point it would have otherwise earned. The response is generally eligible for all subsequent points (unless no answer is possible in degree mode or the question is made simpler by using degree mode). In degree mode, $x_P(1)$ is 5.007 (or 5.006).

Total for part (a) 3 points

(b) Are particles P and Q moving toward each other or away from each other at time $t = 1$? Explain your reasoning.

$v_P(1) = \sin\left(1^{1.5}\right) = 0.841471 > 0$ At time $t = 1$, particle P is moving to the right.	Direction of motion for one particle	**1 point**
$v_Q(1) = (1 - 1.8) \cdot 1.25^1 = -1 < 0$ At time $t = 1$, particle Q is moving to the left. At time $t = 1$, $x_P(1) < x_Q(1)$, so particle P is to the left of particle Q. Thus, at time $t = 1$, particles P and Q are moving toward each other.	Answer with explanation	**1 point**

Scoring notes:

- The first point is earned for using the sign of $v_P(1)$ or $v_Q(1)$ to determine the direction of motion for one of the particles. This point cannot be earned without reference to the sign of $v_P(1)$ or $v_Q(1)$.
- It is not necessary to present an explicit value for $v_P(1)$, or $v_Q(1)$, but if a value is presented, it must be correct as far as reported, up to three places after the decimal.
- Read with imported incorrect position values from part (a).
- If one or both position values were not found in part (a), but are found in part (b), the points for part (a) are not earned retroactively.
- To earn the second point the explanation must be based on the signs of $v_P(1)$ and $v_Q(1)$ and the relative positions of particle P and particle Q at $t = 1$. References to other values of time, such as $t = 0$, are not sufficient.
- Degree mode: $v_P(1) = 0.017$. (See degree mode statement in part (a).)

Total for part (b) 2 points

(c) Find the acceleration of particle Q at time $t = 1$. Is the speed of particle Q increasing or decreasing at time $t = 1$? Explain your reasoning.

$a_Q(1) = v'_Q(1) = 1.026856$ The acceleration of particle Q is 1.027 (or 1.026) at time $t = 1$.	Setup and acceleration	**1 point**
$v_Q(1) = -1 < 0$ and $a_Q(1) > 0$ The speed of particle Q is decreasing at time $t = 1$ because the velocity and acceleration have opposite signs.	Speed decreasing with reason	**1 point**

Scoring notes:

- To earn the first point the acceleration must be explicitly connected to v'_Q (e.g., $v'_Q(1) = 1.026856$).
- The first point is not earned for an unsupported value of 1.027 (or 1.026). The setup, $v'_Q(1)$, must be shown. Presenting only $a_Q(1) = 1.027$ (or 1.026) without indication that $v'_Q = a_Q$ is not enough to earn the first point.
- A response does not need to present a value for $v_Q(1)$; the sign is sufficient.
- To earn the second point a response must compare the signs of a_Q and v_Q at $t = 1$. Considering only one sign is not sufficient.
- After the first point has been earned, a response declaring only "velocity and acceleration are of opposite signs at $t = 1$ so the particle is slowing down" (or equivalent) earns the second point.
- The second point may be earned without the first, as long as the response does not present an incorrect value or sign for $v_Q(1)$ and concludes the particle is slowing down because velocity and acceleration have opposite signs at $t = 1$.

Total for part (c) 2 points

(d) Find the total distance traveled by particle P over the time interval $0 \le t \le \pi$.

$\int_0^{\pi} \lvert v_P(t) \rvert \, dt = 1.93148$ Over the time interval $0 \le t \le \pi$, the total distance traveled by particle P is 1.931.	Definite integral	**1 point**
	Answer	**1 point**

Scoring notes:

- The first point is earned for $\int_0^{\pi} \lvert v_P(t) \rvert \, dt$.
- The first point can also be earned for a sum (or difference) of definite integrals, such as $\int_0^{2.145029} v_P(t) \, dt - \int_{2.145029}^{\pi} v_P(t) \, dt$, provided the response has indicated $v_P(2.145029) = 0$.
- The second point can only be earned for the correct answer.
- The unsupported value 1.931 earns no points.
- A response reporting the distance traveled by particle Q as $\int_0^{\pi} \lvert v_Q(t) \rvert \, dt = 3.506$ earns the first point and is not eligible for the second point.
- In degree mode, the total distance traveled is 0.122. (See degree mode statement in part (a).) In the degree mode case, the response must present $\int_0^{\pi} \lvert v_P(t) \rvert \, dt$ in order to earn the first point because $\int_0^{\pi} \lvert v_P(t) \rvert \, dt = \int_0^{\pi} v_P(t) \, dt$.

Total for part (d) 2 points

Total for question 2 9 points

Answer QUESTION 2 parts (a) and (b) on this page.

Response for question 2(a)

position of $P = X_P = 5 + \int_0^1 \sin(t^{1.5})\,dt = 5 + 0.37066$

Position of $Q = X_Q = 10 + \int_0^1 V_Q(t)\,dt = 8.56435$

At time $t=1$, the position of particle P is 5.37066 and the position of particle Q is 8.56435

Response for question 2(b)

$V_Q(1) = -1$ $X_Q(1) = 5.37066$

$V_P(2) = 0.84147$ $X_P(1) = 8.56435$

At time $t=1$ the 2 particles are moving closer to each other because $X_P(1) = 5.37066$ and $X_Q(1) = 8.56435$, which means that particle Q is to the right of particle P at $t=1$, and since particle Q has a negative velocity it is moving left and particle P has a positive velocity it is moving right, so the particles are moving toward each other at time $t=1$.

Answer QUESTION 2 parts (c) and (d) on this page.

Response for question 2(c)

a_Q = acceleration of particle Q

$a_Q = v_Q'(t)$ $v_Q'(1) = 1.02686.$

$v_Q(1) = -1.$ Since the $a_Q(1)$ is 1.02686 and is positive and and $v_Q(1)$ is negative, the speed of particle Q at time $t=1$ must be decreasing.

Response for question 2(d)

Total distance particle P travelled from $0 \le t \le \pi$ = $\int_0^{\pi} |v_p(t)|\,dt = 1.93148$

The total distance particle P travelled from $0 \le t \le \pi$ is 1.93148.

Answer QUESTION 2 parts (a) and (b) on this page.

Response for question 2(a)

$x_P(1) = 5 + \int_0^1 (\sin(t^{1.5}))\,dx = 5 + 0.371 = 5.371$ units

$x_Q(1) = 10 + \int_0^1 (1.25^t(t - 1.8))\,dx = 10 + (-1.436) = 8.564$ units

Response for question 2(b)

$v_P(1) = 0.841$ $a_P(1) = 0.810$

$v_Q(1) = -1$ $a_Q(1) = 1.027$

At $t = 1$, the particles are moving away from each other as their velocities are going in diffrent directions at $t = 1$

Answer QUESTION 2 parts (c) and (d) on this page.

Response for question 2(c)

$a_Q(t) = v_Q'(t) = 1.25^t \ln 1.25 (t - 1.8) + 1.25^t$

$a_Q(1) = 1.25^{(1)} \ln 1.25 (1 - 1.8) + 1.25^1 = 1.027$ units/t^2

At $t = 1$, the speed of Q is decreasing b/c $a_Q(t)$ which is $v_Q'(t)$ is positive while $v_Q(1)$ is negative.

Response for question 2(d)

$\int_0^{\pi} (|\sin(t^{1.5})|)\,dx = 1.931$

total distance traveled = 1.931

2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2

Answer QUESTION 2 parts (a) and (b) on this page.

Response for question 2(a)

P: $v(t) = \sin(t^{1.5})$

Q: $v(t) = (t - 1.8) \cdot 1.25^t$

$\int_0^1 v_P(t)\,dt = .37066$

$\int_0^1 v_Q(t)\,dt = -1.4356$

Response for question 2(b)

Particles P & Q are moving away from each other at time t=1 because one velocity (P) is positive and one velocity (Q) is negative.

2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2

Answer QUESTION 2 parts (c) and (d) on this page.

Response for question 2(c)

$v(t) = \sin(t^{1.5})$

$v'(t) = a(t)$

$a(1) = 1.0269$

The speed of particle Q is decreasing at time t=1 because the velocity is negative and the accleration is positive, which are different signs. $v(1) < 0$ and $a(1) > 0$.

Response for question 2(d)

$\int_0^{\pi} |v_p(t)|\, dt$

$\int_0^{\pi} |\sin(t^{1.5})|\, dt$

1.9315 = total distance on interval $0 \leq t \leq \pi$

Question 2

Note: Student samples are quoted verbatim and may contain spelling and grammatical errors.

Overview

In this problem particles P and Q move along the x-axis with velocities $v_P(t) = \sin\left(t^{1.5}\right)$ and $v_Q(t) = (t - 1.8) \cdot 1.25^t$, respectively. The velocity of both particles applies for $0 \le t \le \pi$, and at time $t = 0$, particle P is at position $x = 5$, while particle Q is at position $x = 10$.

In part (a) students were asked to find the positions of both particles at time $t = 1$. A correct response should find the net change in each particle's position as the integral of their respective velocity across the interval $0 \le t \le 1$ and add this change to each particle's position at time $t = 1$.

In part (b) students were asked whether the particles were moving toward or away from each other at this time ($t = 1$). A correct response should evaluate the given velocity functions at $t = 1$ to determine the sign of each particle's velocity. This should lead to the conclusion that particle P is moving to the right while particle Q is moving to the left. In addition, a response should use the position functions found in part (a) to determine that at time $t = 1$ particle P is to the left of particle Q and, therefore, the particles are moving toward each other.

In part (c) students were asked to find the acceleration of particle Q at time $t = 1$ and whether the speed of particle Q was increasing or decreasing at time $t = 1$. A correct response should indicate that acceleration is the derivative of velocity and find the value of $a_Q = v_Q'$ at time $t = 1$ using a graphing calculator. The response should then indicate that the particle's speed is decreasing because the particle's acceleration and velocity (sign determined in part (b)) have opposite signs at this time.

Finally, in part (d) students were asked to find the total distance traveled by particle P over the entire time interval $0 \le t \le \pi$. A correct response would use a graphing calculator to determine the value of the definite integral of the speed, $\int_0^{\pi} |v_P(t)|\, dt$.

Sample: 2A
Score: 9

The response earned 9 points: 3 points in part (a), 2 points in part (b), 2 points in part (c), and 2 points in part (d). In part (a) the response earned the first point for $\int_0^1 \sin\left(t^{1.5}\right) dt$ on line 1. The response earned the second point for $5 + \int_0^1 \sin\left(t^{1.5}\right) dt = 5 + 0.37066$ on line 1. Note that the numerical expression $5 + 0.37066$ need not be simplified. The response earned the third point for $10 + \int_0^1 v_Q(t)\, dt = 8.56435$ on line 2. Lines 3-5 summarize the results and contain correct information. Note the presented decimals are accurate to three decimal places, rounded or truncated.

In part (b) the response earned the first point by stating that "since particle Q has a negative velocity it is moving left" on lines 7 and 8. The response earned the second point by stating that "Q is to the right of particle P at $t = 1$," "particle Q has a negative velocity it is moving left," and "particle P has a positive velocity it is moving right, so the particles are moving toward each other at time $t = 1$" on lines 6-10. In part (c) the response earned the first point on line 2 for $a_Q = v_Q{}'(t)$ and $v_Q{}'(1) = 1.02686$. Note that without $a_Q = v_Q{}'(t)$, the response would still have earned the first point for $v_Q{}'(1) = 1.02686$. The response earned the second point for comparing the signs of $a_Q(1)$ (positive) and $v_Q(1)$ (negative) and concluding that the speed of particle Q at time $t = 1$ must be decreasing on

Question 2 (continued)

lines 3-6. In part (d) the response earned the first point for $\int_0^{\pi} |v_P(t)|\, dt$ on line 2. The response earned the second point for $\int_0^{\pi} |v_P(t)|\, dt = 1.93148$ on line 2. Lines 3 and 4 summarize the result and contain correct information. Note the presented decimal is accurate to three decimal places, rounded or truncated.

Sample: 2B
Score: 7

The response earned 7 points: 3 points in part (a), no points in part (b), 2 points in part (c), and 2 points in part (d). In part (a) the response earned the first point for $\int_0^1 \sin\left(t^{1.5}\right) dx$ on line 1. The second point was earned for $5 + \int_0^1 \sin\left(t^{1.5}\right) dx = 5 + 0.371$ on line 1. Note the correct position of particle P need not be simplified; however, the response simplifies correctly to obtain $5.371.$ The third point was earned for $10 + \int_0^1 \left(1.25^t (t - 1.8)\right) dx = 10 + (-1.436) = 8.564$ on line 2. The response was not penalized for the use of dx in place of $dt.$ In part (b) no points were earned because the response fails to connect the direction of motion of each particle with the correct signs of the respective velocities at $t = 1.$ Also, the response fails to reference the relative positions of P and Q at $t = 1.$ In part (c) the first point was earned on lines 1 and 2 of the response. Note that the correct expression for $v_Q'(t)$ is given on line 1; however, this was not required. On line 1 the connection between $a_Q(t)$ and $v_Q'(t)$ is made. Note the required connection is also made if the response begins the statement on line 1 with $v_Q'(t).$ On line 2 the correct value of $a_Q(1)$ is given. The second point was earned on lines 3 and 4 by comparing the signs of v_Q and a_Q at $t = 1$ and concluding that the speed of Q is decreasing. In part (d) the response earned the first point for $\int_0^{\pi} \left(\left|\sin\left(t^{1.5}\right)\right|\right) dx$ on line 1. The response was not penalized for the use of dx in place of $dt.$ The second point was earned for the correct total distance traveled, $\int_0^{\pi} \left(\left|\sin\left(t^{1.5}\right)\right|\right) dx = 1.931,$ on line 1. The response goes on to summarize the result, which is unnecessary but correct, so the response earned the second point on line 2.

Sample: 2C
Score: 5

The response earned 5 points: 1 point in part (a), no points in part (b), 2 points in part (c), and 2 points in part (d). In part (a) the response earned the first point for $\int_0^1 v_P(t)\, dt$ on line 3. The second and third points were not earned because both positions are incorrect. In part (b) the response earned no points because there is no connection made between the direction of motion of either particle and the sign of its velocity. Also, there is no reference to the relative positions of the particles at $t = 1,$ and the response incorrectly concludes that the particles are moving away from each other. In part (c) the first point was earned on lines 2 and 3 where the connection between $v'(t)$ and $a(t)$ is made explicit, and the correct value of $a(1) = 1.0269$ is stated. The second point was earned for "The speed of particle Q is decreasing at time $t = 1$ because the velocity is negative and the acceleration is positive, which are different signs." The response goes on to state the signs of $v(1) < 0$ and $a(1) > 0$ on lines 7 and 8. In part (d) the

Question 2 (continued)

response earned the first point for the definite integral $\int_0^{\pi} |v_P(t)|\, dt$ on line 1. The second point was earned for $\int_0^{\pi} |v_P(t)|\, dt$ and the correct total distance traveled, $1.9315,$ stated on line 3. Note that the stated answer of $1.9315,$ which is given to four decimals places, is correct when truncated to three decimal places.

2021

AP CollegeBoard

AP® Calculus AB

Sample Student Responses and Scoring Commentary

Inside:

Free Response Question 3

☑ Scoring Guideline

☑ Student Samples

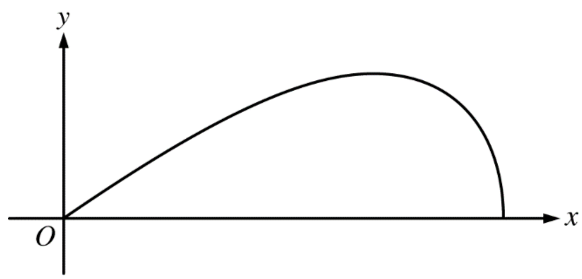
☑ Scoring Commentary

Part B (AB or BC): Graphing calculator not allowed
Question 3 — 9 points

General Scoring Notes

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

Scoring guidelines and notes contain examples of the most common approaches seen in student responses. These guidelines can be applied to alternate approaches to ensure that these alternate approaches are scored appropriately.



A company designs spinning toys using the family of functions $y = cx\sqrt{4 - x^2}$, where c is a positive constant. The figure above shows the region in the first quadrant bounded by the x-axis and the graph of $y = cx\sqrt{4 - x^2}$, for some c. Each spinning toy is in the shape of the solid generated when such a region is revolved about the x-axis. Both x and y are measured in inches.

	Model Solution	Scoring	
(a)	Find the area of the region in the first quadrant bounded by the x-axis and the graph of $y = cx\sqrt{4 - x^2}$ for $c = 6$.		
	$6x\sqrt{4 - x^2} = 0 \Rightarrow x = 0,\ x = 2$ Area $= \int_0^2 6x\sqrt{4 - x^2}\, dx$	Integrand	**1 point**
	Let $u = 4 - x^2$. $du = -2x\, dx \Rightarrow -\frac{1}{2}\, du = x\, dx$ $x = 0 \Rightarrow u = 4 - 0^2 = 4$ $x = 2 \Rightarrow u = 4 - 2^2 = 0$ $\int_0^2 6x\sqrt{4 - x^2}\, dx = \int_4^0 6\left(-\frac{1}{2}\right)\sqrt{u}\, du = -3\int_4^0 u^{1/2}\, du = 3\int_0^4 u^{1/2}\, du$ $= 2u^{3/2}\Big\|_{u=0}^{u=4} = 2 \cdot 8 = 16$	Antiderivative	**1 point**
	The area of the region is 16 square inches.	Answer	**1 point**

Scoring notes:

- Units are not required for any points in this question and are not read if presented (correctly or incorrectly) in any part of the response.
- The first point is earned for presenting $cx\sqrt{4 - x^2}$ or $6x\sqrt{4 - x^2}$ as the integrand in a definite integral. Limits of integration (numeric or alphanumeric) must be presented (as part of the definite integral) but do not need to be correct in order to earn the first point.
- If an indefinite integral is presented with an integrand of the correct form, the first point can be earned if the antiderivative (correct or incorrect) is eventually evaluated using the correct limits of integration.
- The second point can be earned without the first point. The second point is earned for the presentation of a correct antiderivative of a function of the form $Ax\sqrt{4 - x^2}$, for any nonzero constant A. If the response has subsequent errors in simplification of the antiderivative or sign errors, the response will earn the second point but will not earn the third point.
- Responses that use u-substitution and have incorrect limits of integration or do not change the limits of integration from x- to u-values are eligible for the second point.
- The response is eligible for the third point only if it has earned the second point.
- The third point is earned only for the answer 16 or equivalent. In the case where a response only presents an indefinite integral, the use of the correct limits of integration to evaluate the antiderivative must be shown to earn the third point.
- The response cannot correct -16 to $+16$ in order to earn the third point; there is no possible reversal here.

Total for part (a) 3 points

(b) It is known that, for $y = cx\sqrt{4 - x^2}$, $\frac{dy}{dx} = \frac{c(4 - 2x^2)}{\sqrt{4 - x^2}}$. For a particular spinning toy, the radius of the largest cross-sectional circular slice is 1.2 inches. What is the value of c for this spinning toy?

The cross-sectional circular slice with the largest radius occurs where $cx\sqrt{4 - x^2}$ has its maximum on the interval $0 < x < 2$. $\frac{dy}{dx} = \frac{c(4 - 2x^2)}{\sqrt{4 - x^2}} = 0 \Rightarrow x = \sqrt{2}$	Sets $\frac{dy}{dx} = 0$	**1 point**
$x = \sqrt{2} \Rightarrow y = c\sqrt{2}\sqrt{4 - (\sqrt{2})^2} = 2c$ $2c = 1.2 \Rightarrow c = 0.6$	Answer	**1 point**

Scoring notes:

- The first point is earned for setting $\frac{dy}{dx} = 0$, $\frac{c(4 - 2x^2)}{\sqrt{4 - x^2}} = 0$, or $c(4 - 2x^2) = 0$.
- An unsupported $x = \sqrt{2}$ does not earn the first point.
- The second point can be earned without the first point but is earned only for the answer $c = 0.6$ with supporting work.

Total for part (b) 2 points

(c) For another spinning toy, the volume is 2π cubic inches. What is the value of c for this spinning toy?

$\text{Volume} = \int_0^2 \pi\left(cx\sqrt{4-x^2}\right)^2\,dx = \pi c^2\int_0^2 x^2\left(4-x^2\right)dx$	Form of the integrand	**1 point**
	Limits and constant	**1 point**
$= \pi c^2\int_0^2\left(4x^2-x^4\right)dx = \pi c^2\left(\frac{4}{3}x^3-\frac{1}{5}x^5\Big\vert_0^2\right)$	Antiderivative	**1 point**
$= \pi c^2\left(\frac{32}{3}-\frac{32}{5}\right) = \frac{64\pi c^2}{15}$ $\frac{64\pi c^2}{15} = 2\pi \Rightarrow c^2 = \frac{15}{32} \Rightarrow c = \sqrt{\frac{15}{32}}$	Answer	**1 point**

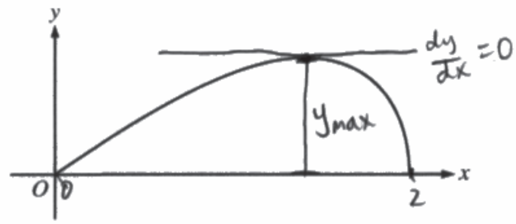
Scoring notes:

- The first point is earned for presenting an integrand of the form $A\left(x\sqrt{4-x^2}\right)^2$ in a definite integral with any limits of integration (numeric or alphanumeric) and any nonzero constant A. Mishandling the c will result in the response being ineligible for the fourth point.
- The second point can be earned without the first point. The second point is earned for the limits of integration, $x = 0$ and $x = 2$, and the constant π (but not for 2π) as part of an integral with a correct or incorrect integrand.
- If an indefinite integral is presented with the correct constant π, the second point can be earned if the antiderivative (correct or incorrect) is evaluated using the correct limits of integration.
- A response that presents $2 = \int_0^2\left(cx\sqrt{4-x^2}\right)^2 dx$ earns the first and second points.
- The third point is earned for presenting a correct antiderivative of the presented integrand of the form $A\left(x\sqrt{4-x^2}\right)^2$ for any nonzero A. If there are subsequent errors in simplification of the antiderivative, linkage errors, or sign errors, the response will not earn the fourth point.
- The fourth point cannot be earned without the third point. The fourth point is earned only for the correct answer. The expression does not need to be simplified to earn the fourth point.

Total for part (c) 4 points

Total for question 3 9 points

Answer QUESTION 3 part (a) on this page.



Response for question 3(a)

$y = 6x\sqrt{4-x^2} = 0$

$x=0 \ , \ x=2$

$A = \int_0^2 6x\sqrt{4-x^2}\,dx$ $\quad u = 4-x^2$, $du = -2x\,dx$

$A = \int_4^0 -3\sqrt{u}\,du = 3\int_0^4 u^{\frac{1}{2}}\,du = 3u^{3/2}\cdot\frac{2}{3}\Big|_0^4$

$A = 2(4^{3/2} - 0^{3/2}) = 2(2^{3} - 0) = 2(8) = 16$

$A = 16$

 Sample 3A

Answer QUESTION 3 parts (b) and (c) on this page.

Response for question 3(b)

Largest cross section when y is greatest (maximum of y on graph).

Find max:

$$\frac{dy}{dx} = \frac{c(4-2x^2)}{\sqrt{4-x^2}} = 0 \rightarrow c(4-2x^2)=0$$
$$4 = 2x^2$$
$$x = \sqrt{2}$$

At $x=\sqrt{2}$, $y=1.2$ (largest radius of cross-section equals 1.2, which is max y value)

$$y = cx\sqrt{4-x^2}$$
$$1.2 = c\sqrt{2}(\sqrt{4-(\sqrt{2})^2}) = c\sqrt{2}(\sqrt{4-2}) = c\sqrt{2}(\sqrt{2}) = 2c$$
$$c = \frac{1.2}{2} = 0.6 \rightarrow \boxed{c = 0.6}$$

Response for question 3(c)

$$V = \pi\int_0^2 y^2\,dx = \pi\int_0^2 (cx\sqrt{4-x^2})^2\,dx = \pi c^2\int_0^2 x^2(4-x^2)\,dx = \pi c^2\int_0^2 (4x^2-x^4)\,dx$$

$$V = \pi c^2\left[\frac{4x^3}{3} - \frac{x^5}{5}\Big|_0^2\right] = \pi c^2\left[\left(\frac{4(8)}{3} - \frac{(32)}{5}\right) - (0-0)\right]$$

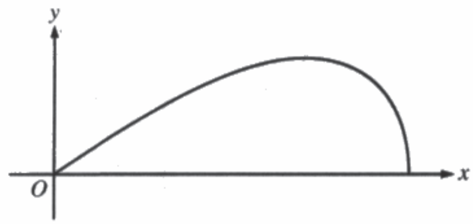
$$V = \pi c^2\left(\frac{32(5)}{3(5)} - \frac{32(3)}{5(3)}\right) = \pi c^2\left(\frac{2(32)}{15}\right) = \pi c^2\left(\frac{64}{15}\right)$$

$$2\pi = \pi c^2\left(\frac{64}{15}\right)$$

$$c^2 = \frac{30}{64} \rightarrow c = \sqrt{\frac{30}{64}} = \boxed{\frac{\sqrt{30}}{8}}$$

3 3 3 3 3 NO CALCULATOR ALLOWED 3 3 3 3 3

Answer QUESTION 3 part (a) on this page.



Response for question 3(a)

$y = 6x\sqrt{4-x^2}$

$6x\sqrt{4-x^2} = 0$

$x = 0$

$x = 2$

$A = \int_0^2 6x\sqrt{4-x^2}\,dx$

$u = 4-x^2$

$du = -2x\,dx$

$-3du = 6x\,dx$

$u = 4-2^2$, $u = 0$

$u = 4-0^2$, $u = 4$

$A = \int_4^0 -3\sqrt{u}\,du$

$A = -3\int_4^0 \sqrt{u}\,du$

$\frac{1}{2} + \frac{2}{2} = \frac{3}{2}$

$-3\left[\frac{2(u)^{3/2}}{3}\right]\Big|_4^0$

$-3\left[\frac{2(4-x^2)^{3/2}}{3}\right]\Big|_0^2$

$= -3\left[\frac{2(4-2^2)^{3/2}}{3} - \frac{2(4-0^2)^{3/2}}{3}\right]$

$= -3\left(0 - \frac{2(4)^{3/2}}{3}\right)$

$= \frac{6(4)^{3/2}}{3} = 2(4)^{3/2}$

$= 16$

3 3 3 3 3 NO CALCULATOR ALLOWED 3 3 3 3 3

Answer QUESTION 3 parts (b) and (c) on this page.

Response for question 3(b)

$\frac{dy}{dx} = 0$

$\frac{c(4-2x^2)}{\sqrt{4-x^2}} = 0$

$c(4-2x^2) = 0$

$4-2x^2=0$

$4=2x^2$

$2=x^2$

$x=\sqrt{2}$

$1.2 = c\sqrt{2} \times \sqrt{4-(\sqrt{2})^2}$

$= c\sqrt{2} \times \sqrt{4-2}$

$= c\sqrt{2} \times \sqrt{2}$

$1.2 = 2c$

$c = \frac{1.2}{2}$

$c = 0.6$

Response for question 3(c)

$V = \pi \int r^2 dx$

$V = \pi \int_0^2 cx\sqrt{4-x^2}\, dx$

$c\pi \int_0^2 x\sqrt{4-x^2}\, dx = 2\pi$

$c\pi \int_4^0 \sqrt{u}\, du = 2\pi$

$c\pi \left[\frac{2u^{3/2}}{3} \Big|_4^0 \right] = 2\pi$

$u = 4-x^2$

$\frac{du}{-2} = \frac{-2x}{-2} dx$

$du = x\, dx$

$u = 4-2^2 = 0$

$4-0^2 = 4$

$\frac{2(4-x^2)^{3/2}}{3} \Big|_0^2 = \frac{2}{c}$

$0 - \frac{2(4)^{3/2}}{3} = \frac{2}{c}$

$\frac{-16}{3} = \frac{2}{c}$

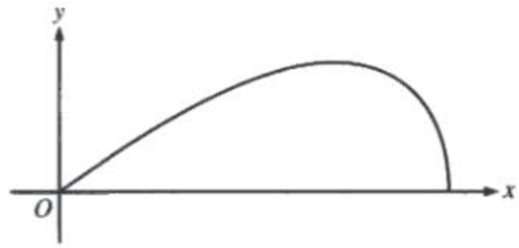
$c = \frac{2\times3}{-16}$

$c = \frac{-6}{16}$

$c = \frac{-3}{8}$

3 3 3 3 3 NO CALCULATOR ALLOWED 3 3 3 3 3

Answer QUESTION 3 part (a) on this page.



Response for question 3(a)

$y = 6x\sqrt{4-x^2}$ $\quad y = 0 = 6x\sqrt{4-x^2}$

$A = \int_0^2 6x(4-x^2)^{1/2}\,dx \quad x = 0, 2$

$A = \frac{1}{2}6\int_0^2 (2x)(4-x^2)^{1/2}\,dx$

$A = 3\left[\frac{2}{3}(4-x^2)^{3/2}\right]\Big|_0^2$

$A = 3\left(\frac{2}{3}(0) - \frac{2}{3}(4)^{3/2}\right)$

$3\left(0 - \frac{16}{3}\right)$

$|-16| = \boxed{16 \text{ inches}^2}$

Answer QUESTION 3 parts (b) and (c) on this page.

Response for question 3(b)

$y = Cx\sqrt{4-x^2}$ $\qquad \frac{dy}{dx} = \frac{C(4-2x^2)}{\sqrt{4-x^2}} = 0$

$4 = 2x^2$

$2 = x^2$

$x = \sqrt{2}$ CN $\qquad \frac{dy}{dx}$: $+++++++$ | $----$ at $\sqrt{2}$

$y(\sqrt{2}) = 1.2 = C\sqrt{2}\sqrt{4-2}$

$1.2 = C\sqrt{2}\sqrt{2}$

$1.2 = 2C$

$\boxed{C = .6}$

Response for question 3(c)

$y = Cx\sqrt{4-x^2} = 0$

$x = 0, 2$

$V = 2\pi \int_0^2 x\left(Cx\sqrt{4-x^2}\right) dx$

$V = 2\pi = 2\pi \int_0^2 x\left(Cx\sqrt{4-x^2}\right) dx$

$\int_0^2 x\left(Cx\sqrt{4-x^2}\right) dx =$

?

Question 3

Note: Student samples are quoted verbatim and may contain spelling and grammatical errors.

Overview

In this problem a company designs spinning toys using various functions of the form $y = cx\sqrt{4 - x^2}$, where c is a positive constant. A graph of the region in the first quadrant bounded by the x-axis and this function for some c is given and students were told that the spinning toys are in the shape of the solid generated when this region is revolved around the x-axis. Both x and y are measured in inches.

In part (a) students were asked to find the area of the region in the first quadrant bounded by the x-axis and the region $y = cx\sqrt{4 - x^2}$ for $c = 6$. A correct response will set up the definite integral $\int_0^2 6x\sqrt{4 - x^2}\,dx$ and use the method of substitution to evaluate the integral to obtain an area of 16.

In part (b) students were told that for $y = cx\sqrt{4 - x^2}$, $\frac{dy}{dx} = \frac{c\left(4 - 2x^2\right)}{\sqrt{4 - x^2}}$. They were also told that for a particular spinning toy the radius of the largest cross-sectional circular slice is 1.2 inches and were asked to find the value of c for this particular spinning toy. A correct response will solve $\frac{dy}{dx} = 0$ to find that the largest radius occurs when $x = \sqrt{2}$. Then using this value of x in the equation $y = cx\sqrt{4 - x^2} = 1.2$, the value of c is found to be 0.6.

In part (c) students were told that for another spinning toy, the volume is 2π cubic inches. They were asked to find the value of c for this spinning toy. A correct response would set up the volume of the toy as the integral $\int_0^2 \pi\left(cx\sqrt{4 - x^2}\right)^2 dx$, evaluate this integral, and set the value equal to 2π. Solving the resulting equation for c results in $c = \sqrt{\frac{15}{32}}$.

Sample: 3A
Score: 9

The response earned 9 points: 3 points in part (a), 2 points in part (b), and 4 points in part (c). In part (a) the response presents the correct integrand in a definite integral and earned the first point. The antiderivative of $3u^{3/2} \cdot \frac{2}{3}$ with the definition $u = 4 - x^2$ is correct and earned the second point. The response has the correct answer and earned the third point. In part (b) the response earned the first point for stating $\frac{dy}{dx} = \frac{c\left(4 - 2x^2\right)}{\sqrt{4 - x^2}} = 0$. The answer is correct, and the response earned the second point. In part (c) the response presents y^2 as the integrand of a definite integral and earned the first point. Note that because $y = cx\sqrt{4 - x^2}$ is given in the statement of the problem, a response can reference the function by using y for the first point. The limits and constant are correct and earned the second point. The antiderivative is correct and earned the third point. The response is eligible for the fourth point. The answer is correct and earned the fourth point. Note that $\frac{\sqrt{30}}{8} = \sqrt{\frac{15}{32}}$.

Question 3 (continued)

Sample: 3B
Score: 6

The response earned 6 points: 3 points in part (a), 2 points in part (b), and 1 point in part (c). In part (a) the response presents the correct integrand in a definite integral and earned the first point. The antiderivative $-3\left[\frac{2(u)^{3/2}}{3}\right]$ with the definition $u = 4 - x^2$ is correct and earned the second point. Note that the sign of the antiderivative is consistent with the limits of integration. The response is eligible for the third point. The answer is correct and earned the third point. Note that the substitution of $u = 4 - x^2$ after finding the antiderivative and using the limits of $x = 0$ and $x = 2$ is not necessary to evaluate the antiderivative. In part (b) the response earned the first point by stating $\frac{dy}{dx} = 0.$ The answer is correct and earned the second point. In part (c) the integrand is not of the correct form and the response did not earn the first point. The limits and constant are correct and earned the second point. Because the integrand is not of the correct form, the response is not eligible for and did not earn the third point. Without earning the third point, the response is not eligible for and did not earn the fourth point.

Sample: 3C
Score: 3

The response earned 3 points: 1 point in part (a), 2 points in part (b), and no points in part (c). In part (a) the response presents the correct integrand in a definite integral and earned the first point. The antiderivative presented is incorrect because the sign of the antiderivative is incorrect, and the response did not earn the second point. The response is not eligible for and did not earn the third point. In part (b) the response earned the first point by stating $\frac{dy}{dx} = \frac{c\left(4 - 2x^2\right)}{\sqrt{4 - x^2}} = 0.$ The answer is correct, and the response earned the second point. In part (c) the integrand presented is not of the correct form and the response did not earn the first point. The constant 2π is incorrect and the response did not earn the second point. Without an integrand of the correct form, the response is not eligible for the third point and is not eligible for the fourth point. The response did not earn the third point and did not earn the fourth point.

2021

AP CollegeBoard

AP® Calculus AB

Sample Student Responses and Scoring Commentary

Inside:

Free Response Question 4

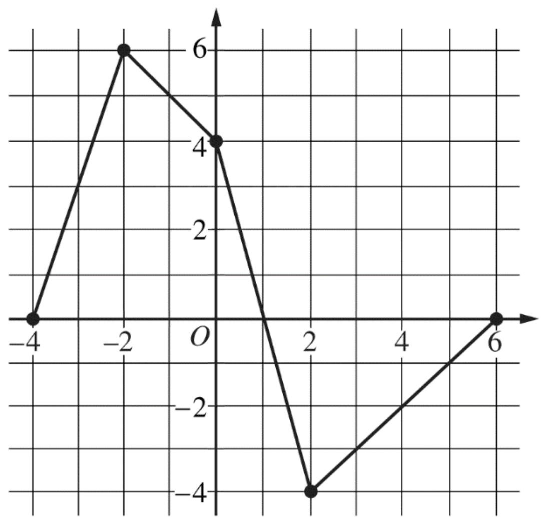
- ☑ Scoring Guideline
- ☑ Student Samples
- ☑ Scoring Commentary

Part B (AB or BC): Graphing calculator not allowed
Question 4 — 9 points

General Scoring Notes

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

Scoring guidelines and notes contain examples of the most common approaches seen in student responses. These guidelines can be applied to alternate approaches to ensure that these alternate approaches are scored appropriately.



Graph of f

Let f be a continuous function defined on the closed interval $-4 \le x \le 6$. The graph of f, consisting of four line segments, is shown above. Let G be the function defined by $G(x) = \int_0^x f(t)\, dt$.

Model Solution	Scoring	
$G'(x) = f(x)$ in any part of the response.	$G'(x) = f(x)$	**1 point**

Scoring notes:

- This “global point” can be earned in any one part. Expressions that show this connection and therefore earn this point include: $G' = f$, $G'(x) = f(x)$, $G''(x) = f'(x)$ in part (a), $G'(3) = f(3)$ in part (b), or $G'(2) = f(2)$ in part (c).

Total **1 point**

(a) On what open intervals is the graph of G concave up? Give a reason for your answer.

$G'(x) = f(x)$ The graph of G is concave up for $-4 < x < -2$ and $2 < x < 6$, because $G' = f$ is increasing on these intervals.	Answer with reason	**1 point**

Scoring notes:

- Intervals may also include one or both endpoints.

Total for part (a) **1 point**

(b) Let P be the function defined by $P(x) = G(x) \cdot f(x)$. Find $P'(3)$.

$P'(x) = G(x) \cdot f'(x) + f(x) \cdot G'(x)$ $P'(3) = G(3) \cdot f'(3) + f(3) \cdot G'(3)$	Product rule	**1 point**
Substituting $G(3) = \int_0^3 f(t)\,dt = -3.5$ and $G'(3) = f(3) = -3$ into the above expression for $P'(3)$ gives the following:	$G(3)$ or $G'(3)$	**1 point**
$P'(3) = -3.5 \cdot 1 + (-3) \cdot (-3) = 5.5$	Answer	**1 point**

Scoring notes:

- The first point is earned for the correct application of the product rule in terms of x or in the evaluation of $P'(3)$. Once earned, this point cannot be lost.
- The second point is earned by correctly evaluating $G(3) = -3.5$, $G'(3) = -3$, or $f(3) = -3$.
- To be eligible to earn the third point a response must have earned the first two points.
- Simplification of the numerical value is not required to earn the third point.

Total for part (b) **3 points**

(c) Find $\lim_{x \to 2} \frac{G(x)}{x^2 - 2x}$.

$\lim_{x \to 2} \left(x^2 - 2x\right) = 0$ Because G is continuous for $-4 \le x \le 6$, $\lim_{x \to 2} G(x) = \int_0^2 f(t)\, dt = 0$. Therefore, the limit $\lim_{x \to 2} \frac{G(x)}{x^2 - 2x}$ is an indeterminate form of type $\frac{0}{0}$.	Uses L'Hospital's Rule	**1 point**
Using L'Hospital's Rule, $\lim_{x \to 2} \frac{G(x)}{x^2 - 2x} = \lim_{x \to 2} \frac{G'(x)}{2x - 2}$ $= \lim_{x \to 2} \frac{f(x)}{2x - 2} = \frac{f(2)}{2} = \frac{-4}{2} = -2$	Answer with justification	**1 point**

Scoring notes:

- To earn the first point the response must show $\lim_{x \to 2} \left(x^2 - 2x\right) = 0$ and $\lim_{x \to 2} G(x) = 0$ and must show a ratio of the two derivatives, $G'(x)$ and $2x - 2$. The ratio may be shown as evaluations of the derivatives at $x = 2$, such as $\frac{G'(2)}{2}$.
- To earn the second point the response must evaluate correctly with appropriate limit notation. In particular the response must include $\lim_{x \to 2} \frac{G'(x)}{2x - 2}$ or $\lim_{x \to 2} \frac{f(x)}{2x - 2}$.
- With any linkage errors (such as $\frac{G'(x)}{2x - 2} = \frac{f(2)}{2}$), the response does not earn the second point.

Total for part (c) **2 points**

(d) Find the average rate of change of G on the interval $[-4, 2]$. Does the Mean Value Theorem guarantee a value c, $-4 < c < 2$, for which $G'(c)$ is equal to this average rate of change? Justify your answer.

$G(2) = \int_0^2 f(t)\,dt = 0$ and $G(-4) = \int_0^{-4} f(t)\,dt = -16$ Average rate of change $= \dfrac{G(2) - G(-4)}{2 - (-4)} = \dfrac{0 - (-16)}{6} = \dfrac{8}{3}$	Average rate of change	**1 point**
Yes, $G'(x) = f(x)$ so G is differentiable on $(-4, 2)$ and continuous on $[-4, 2]$. Therefore, the Mean Value Theorem applies and guarantees a value c, $-4 < c < 2$, such that $G'(c) = \frac{8}{3}$.	Answer with justification	**1 point**

Scoring notes:

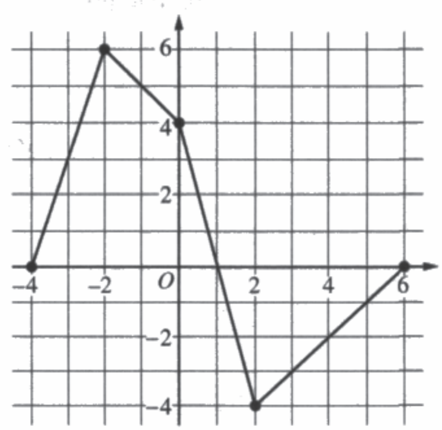
- To earn the first point a response must present at least a difference and a quotient and a correct evaluation. For example, $\frac{0 + 16}{6}$ or $\frac{G(2) - G(-4)}{6} = \frac{16}{6}$.
- Simplification of the numerical value is not required to earn the first point, but any simplification must be correct.
- The second point can be earned without the first point if the response has the correct setup but an incorrect or no evaluation of the average rate of change. The Mean Value Theorem need not be explicitly stated provided the conditions and conclusion are stated.

Total for part (d) 2 points

Total for question 4 9 points

4 4 4 4 4 NO CALCULATOR ALLOWED 4 4 4 4 4

Answer QUESTION 4 parts (a) and (b) on this page.



Graph of f

Response for question 4(a)

$G'(x) = f(x)$

$G''(x) = f'(x)$

On $(-4,-2)$ and $(2,6)$, $G(x)$ is concave up because $f(x)$ (which is equal to $G'(x)$) has a positive slope / is increasing.

Response for question 4(b)

$P'(x) = G'(x)\,f(x) + f'(x)\,G(x)$

$P'(3) = G'(3)\,f(3) + f'(3)\,G(3)$

$G'(x) = f(x)$

$G'(3) = f(3) = -3$

$G(3) = \int_0^3 f(t)\,dt = -\frac{7}{2}$

$P'(3) = (-3)(-3) + (1)\left(-\frac{7}{2}\right)$

Sample 4A

Answer QUESTION 4 parts (c) and (d) on this page.

Response for question 4(c)

$\lim_{x\to 2} G(x) = \lim_{x\to 2} (x^2-2x) = 0$ Must use l'hôpital's rule

$\llcorner \int_0^2 f(t)\,dt = 0$

$\lim_{x\to 2} \frac{G'(x)}{2x-2} = \frac{f(2)}{4-2} = \frac{-4}{2}$

Response for question 4(d)

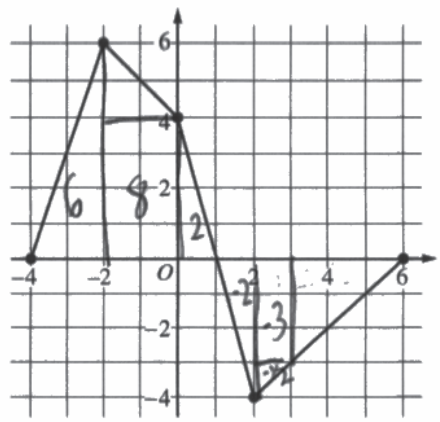
$\int_0^2 f(t)\,dt \quad \int_0^{-4} f(t)\,dt = -(3+9+3+1) = -16$

$\text{AROC of } G = \frac{G(2)-G(-4)}{2-(-4)} = \frac{0-(-16)}{2+4} = \frac{16}{6} = \frac{8}{3}$

The meanvalue theorem does guarantee a value c, $-4 < c < 2$, for which $G'(c)$ is equal to average rate of change. This is because ~~$G'(x) = f(t)$~~ and ~~$x=t$~~ exists for all values, ~~$-4<x=t<2$~~, meaning that $G(x)$ is continuous on the closed interval and differentiable on the open interval.

4 4 4 4 4 NO CALCULATOR ALLOWED 4 4 4 4 4

Answer QUESTION 4 parts (a) and (b) on this page.



Graph of f

Response for question 4(a)

The graph of G is concave up on the intervals (-4, -2) ∪ (2, 6) because G is concave up when G' is increasing and G'(x) = f(x).

Response for question 4(b)

$(4)(1)(\frac{1}{2}) = \cancel{2} - 2 - 3 - \frac{1}{2} = -3 - \frac{1}{2} = -\frac{6}{2} - \frac{1}{2} = -\frac{5}{2}$

$P(x) = G(x) \cdot f(x)$

$P'(x) = G'(x) \cdot f(x) + f'(x) \cdot G(x)$

$P'(3) = G'(3) \cdot f(3) + f'(3) \cdot G(3)$

$= (-3) \cdot (-3) + (-4) \cdot (-\frac{5}{2})$

$= 9 + 10 = 19$

$-\frac{5}{2} \cdot -\frac{4}{1} = 10$

$P'(3) \approx 19$

Answer QUESTION 4 parts (c) and (d) on this page.

Response for question 4(c)

$\lim_{x \to 2} \frac{G(x) \to 0}{x^2 - 2x \to 0}$ L'hopital rule

$\lim_{x \to 2} \frac{G'(x)}{2x - 2} = \frac{G'(2)}{2(2)-2} = \frac{-4}{2} = -2$

Response for question 4(d)

$\frac{1}{6} \int_{-4}^{2} -4 = -\frac{4}{6} = -\frac{2}{3}$

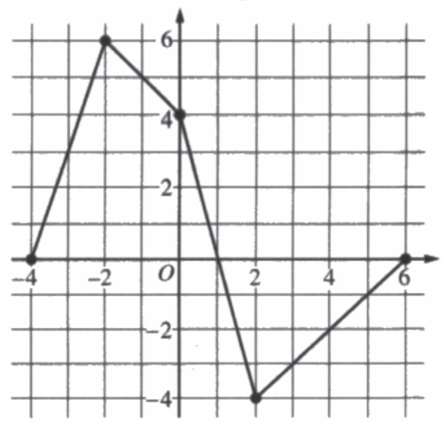
$\frac{G(2) - G(-4)}{-4-2} = \frac{0 - (-(6+8+2))}{-6} = \frac{16}{-6} = -\frac{8}{3}$

The MVT doesn't guarantee the value of c, $-4 < c < 2$ for which $G'(c)$ is equal to the average rate of change of G because the values of the average rate of change are different since G' is the derivative G,

$\frac{G'(2) - G'(-4)}{-4-2} = \frac{-4-0}{-6} = \frac{-4}{-6} = \frac{2}{3}$ $\frac{1}{2-4} \int_{-4}^{2} G(x)dx = \frac{1}{6} \cdot 16 = \frac{16}{6} = \frac{8}{3}$

4 4 4 4 4 NO CALCULATOR ALLOWED 4 4 4 4 4

Answer QUESTION 4 parts (a) and (b) on this page.



Graph of f

Response for question 4(a)

$G(x) = \int_0^{x} f(t)\,dt$

$G(x) = \int f(x)\,dx$

$f(x) = G'(x)$

Because f(x) is the antiderivative of G(x) that means that f(x) = G'(x). Therefore when the graph f(x) = increasing, then that means G(x) = concave up. G is concave up on the intervals (−4, −2) ∪ (2, 6) b/c f is increasing.

Response for question 4(b)

$P(x) = G(x) \cdot f(x)$

$P'(x) = G'(x) \cdot f(x) + f'(x) \cdot G(x)$

$P'(3) = G'(2) \cdot f(2) + f'(3) \cdot G(2)$

$P'(3) = -3 \cdot -3 + -3 \cdot -3$

$P'(3) = 9 + 9$

$P'(3) = 18$

Answer QUESTION 4 parts (c) and (d) on this page.

Response for question 4(c)

$$\lim_{x\to 2} \frac{G(x)}{x^2-2x} \overset{\text{L'Hop}}{\rightarrow} \frac{G'(x)}{2x-2} = \frac{f(x)}{2x-2}$$

$G'(x) = f(x)$

$$= \frac{f(2)}{2(2)-2}$$

$$= \frac{-4}{2}$$

$$= -2$$

Response for question 4(d)

$$G(x) = \int_{-4}^{2} f(x)\,dx$$

$$G'(x) = f(x)\Big|_{-4}^{2}$$

$$= f(2) - f(-4)$$

$$= -4 - 0$$

$$= \boxed{-4}$$

$-4 < c < 2 \quad G'(c) = -4?$

NO b/c c cannot equal -4, it can only be greater than it. Therefore the Mean Value Theorem does not guarantee a value.

Question 4

Note: Student samples are quoted verbatim and may contain spelling and grammatical errors.

Overview

In this problem the graph of a piecewise linear continuous function f for $-4 \le x \le 6$ is provided. It is also given that $G(x) = \int_0^x f(t)\,dt.$

In part (a) students were asked to provide the open intervals on which the graph of G is concave up. A correct response would use the Fundamental Theorem of Calculus to note that $G' = f,$ and then report the two intervals where $G' = f$ is increasing.

In part (b) the function $P(x) = G(x) \cdot f(x)$ is defined and students were asked to find $P'(3).$ A correct response would use the product rule to find an expression for $P'(x),$ then use the graph of f to find numerical values of $f(3)$ and $f'(3),$ and use the Fundamental Theorem of Calculus to find $G(3)$ and $G'(3).$ The response would substitute these values into the expression for $P'(x)$ to provide the value of $P'(3).$

In part (c) students were asked to find $\lim_{x \to 2} \frac{G(x)}{x^2 - 2x}.$ A correct response would use L'Hospital's Rule to find the limit after verifying that the limits of both the numerator and denominator are zero.

In part (d) students were asked to find the average rate of change of G on the interval $[-4, 2]$ and whether the Mean Value Theorem guarantees a value $c,$ $-4 < c < 2,$ with $G'(c)$ equal to this average rate of change. A correct response would determine the average rate of change as a difference quotient, $\frac{G(2) - G(-4)}{2 - (-4)},$ with values $G(2) = 0$ and $G(-4) = -16$ found as areas under the graph of $f.$ The response should then conclude that the Mean Value Theorem does guarantee such a value of c because $G' = f$ is differentiable and, therefore, continuous on the given interval.

Sample: 4A
Score: 9

The response earned 9 points: 1 global point, 1 point in part (a), 3 points in part (b), 2 points in part (c), and 2 points in part (d). The global point was earned in the first line of part (a) with the statement $G'(x) = f(x).$ In part (a) the response earned the point with the presentation of the correct intervals and the reason "$f(x)$ which is equal to $G'(x)$ has a positive slope/is increasing." In part (b) the response earned the first point with the correct product rule presentation in the first line. The second point was earned for the correct values for both $G'(3)$ and $G(3)$ although only one correct value was necessary for this point. The third point was earned with the expression $(-3)(-3)+(1)\left(-\frac{7}{2}\right).$ Simplification of this expression is not necessary. In part (c) the response earned the first point with the extended equation of limits in the first line and the ratio of derivatives in the second line. The second point was earned with the correct ratio of derivatives accompanied by limit notation and the correct (unsimplified) answer. In part (d) the response earned the first point for a valid attempt to calculate the average rate of change of G and a correct result. The second point was earned with the answer, "The mean value theorem does guarantee a value c," and the statement that $G(x)$ is both differentiable and continuous.

Question 4 (continued)

Sample: 4B
Score: 6

The response earned 6 points: 1 global point, 1 point in part (a), 2 points in part (b), 2 points in part (c), and no points in part (d). The global point was earned in part (a) with the statement $G'(x) = f(x)$. In part (a) the response earned the point with the presentation of the correct intervals and the reason that G' is increasing. In part (b) the response earned the first point with the correct product rule in the second line. The second point was earned with the value of $G'(3)$ as -3 in the fourth line. Note that the incorrect value of $G(3)$ did not affect this point because only one correct value of $G(3)$ or $G'(3)$ is required. The third point was not earned due to the incorrect final answer. In part (c) the first point was earned with the arrows pointing from the numerator and denominator to the value 0 and by the ratio of derivatives. The second point was earned with the correct ratio of derivatives accompanied by limit notation and the correct answer. In part (d) the first point was not earned because the average rate of change presented is not correct (denominator should be $2-(-4)$). Because this is not a valid average rate of change form, the response is not eligible for the second point.

Sample: 4C
Score: 4

The response earned 4 points: 1 global point, 1 point in part (a), 2 points in part (b), no points in part (c), and no points in part (d). The global point was earned in the third line of part (a) with the statement $f(x) = G'(x)$. In part (a) the response earned the point with the presentation of the correct intervals and the reason "when $f(x) =$ increasing, then $G(x) =$ concave up." In part (b) the response earned the first point with the correct product rule in the second line. The second point was earned for the correct value for $G(3)$. The value for $G'(3)$ is incorrect, but only one correct value is necessary for this point. The third point was not earned because the final answer is incorrect. In part (c) the response did not earn the first point because there is no evidence of $\lim_{x\to 2} G(x) = 0$ or $\lim_{x\to 2}\left(x^2 - 2x\right) = 0$ given. The second point was not earned because the ratio of derivatives does not have limit notation. In part (d) the response did not earn the first point because there is not an attempt to calculate the average rate of change of G. The response is not eligible for the second point.

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Sample Student Responses and Scoring Commentary

Inside:

Free Response Question 5

☑ Scoring Guideline

☑ Student Samples

☑ Scoring Commentary

Part B (AB): Graphing calculator not allowed
Question 5 — 9 points

General Scoring Notes

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

Scoring guidelines and notes contain examples of the most common approaches seen in student responses. These guidelines can be applied to alternate approaches to ensure that these alternate approaches are scored appropriately.

Consider the function $y = f(x)$ whose curve is given by the equation $2y^2 - 6 = y \sin x$ for $y > 0$.

	Model Solution	Scoring	
(a)	Show that $\frac{dy}{dx} = \frac{y \cos x}{4y - \sin x}$.		
	$\frac{d}{dx}\left(2y^2 - 6\right) = \frac{d}{dx}(y \sin x) \Rightarrow 4y\frac{dy}{dx} = \frac{dy}{dx}\sin x + y \cos x$	Implicit differentiation	**1 point**
	$\Rightarrow 4y\frac{dy}{dx} - \frac{dy}{dx}\sin x = y \cos x \Rightarrow \frac{dy}{dx}(4y - \sin x) = y \cos x$ $\Rightarrow \frac{dy}{dx} = \frac{y \cos x}{4y - \sin x}$	Verification	**1 point**

Scoring notes:

- The first point is earned only for correctly implicitly differentiating $2y^2 - 6 = y \sin x$. Responses may use alternative notations for $\frac{dy}{dx}$, such as y'.
- The second point may not be earned without the first point.
- It is sufficient to present $\frac{dy}{dx}(4y - \sin x) = y \cos x$ to earn the second point, provided that there are no subsequent errors.

Total for part (a) 2 points

(b) Write an equation for the line tangent to the curve at the point $(0, \sqrt{3})$.

At the point $(0, \sqrt{3})$, $\frac{dy}{dx} = \frac{\sqrt{3}\cos 0}{4\sqrt{3} - \sin 0} = \frac{1}{4}$. An equation for the tangent line is $y = \sqrt{3} + \frac{1}{4}x$.	Answer	**1 point**

Scoring notes:

- Any correct tangent line equation will earn the point. No supporting work is required. Simplification of the slope value is not required.

Total for part (b) 1 point

(c) For $0 \le x \le \pi$ and $y > 0$, find the coordinates of the point where the line tangent to the curve is horizontal.

$\frac{dy}{dx} = \frac{y\cos x}{4y - \sin x} = 0 \Rightarrow y\cos x = 0$ and $4y - \sin x \ne 0$	Sets $\frac{dy}{dx} = 0$	**1 point**
$y\cos x = 0$ and $y > 0 \Rightarrow x = \frac{\pi}{2}$	$x = \frac{\pi}{2}$	**1 point**
When $x = \frac{\pi}{2}$, $y\sin x = 2y^2 - 6 \Rightarrow y\sin\frac{\pi}{2} = 2y^2 - 6$ $\Rightarrow y = 2y^2 - 6 \Rightarrow 2y^2 - y - 6 = 0$ $\Rightarrow (2y+3)(y-2) = 0 \Rightarrow y = 2$ When $x = \frac{\pi}{2}$ and $y = 2$, $4y - \sin x = 8 - 1 \ne 0$. Therefore, the line tangent to the curve is horizontal at the point $\left(\frac{\pi}{2}, 2\right)$.	$y = 2$	**1 point**

Scoring notes:

- The first point is earned by any of $\frac{dy}{dx} = 0$, $\frac{y\cos x}{4y - \sin x} = 0$, $y\cos x = 0$, or $\cos x = 0$.
- If additional "correct" x-values are considered outside of the given domain, the response must commit to only $x = \frac{\pi}{2}$ to earn the second point. Any presented y-values, correct or incorrect, are not considered for the second point.
- Entering with $x = \frac{\pi}{2}$ does not earn the first point, earns the second point, and is eligible for the third point. The third point is earned for finding $y = 2$. The coordinates do not have to be presented as an ordered pair.
- The third point is not earned with additional points present unless the response commits to the correct point.

Total for part (c) 3 points

(d) Determine whether f has a relative minimum, a relative maximum, or neither at the point found in part (c). Justify your answer.

$\frac{d^2y}{dx^2} = \frac{(4y - \sin x)\left(\frac{dy}{dx}\cos x - y\sin x\right) - (y\cos x)\left(4\frac{dy}{dx} - \cos x\right)}{(4y - \sin x)^2}$	Considers $\frac{d^2y}{dx^2}$	**1 point**
When $x = \frac{\pi}{2}$ and $y = 2$, $\frac{d^2y}{dx^2} = \frac{\left(4\cdot 2 - \sin\frac{\pi}{2}\right)\left(0\cdot\cos\frac{\pi}{2} - 2\cdot\sin\frac{\pi}{2}\right) - \left(2\cos\frac{\pi}{2}\right)\left(4\cdot 0 - \cos\frac{\pi}{2}\right)}{\left(4\cdot 2 - \sin\frac{\pi}{2}\right)^2}$ $= \frac{(7)(-2) - (0)(0)}{(7)^2} = \frac{-2}{7} < 0.$	$\frac{d^2y}{dx^2}$ at $\left(\frac{\pi}{2}, 2\right)$	**1 point**
f has a relative maximum at the point $\left(\frac{\pi}{2}, 2\right)$ because $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} < 0$.	Answer with justification	**1 point**

Scoring notes:

- The first point is earned for an attempt to use the quotient rule (or product rule) to find $\frac{d^2y}{dx^2}$.
- The second point is earned for correctly finding $\frac{d^2y}{dx^2}$ and evaluating to find that $\frac{d^2y}{dx^2} < 0$ at $\left(\frac{\pi}{2}, 2\right)$. The explicit value of $-\frac{2}{7}$ or the equivalent does not need to be reported, but any reported values must be correct in order to earn this point.
- The third point can be earned without the second point by reaching a consistent conclusion based on the reported sign of a nonzero value of $\frac{d^2y}{dx^2}$ obtained utilizing $\frac{dy}{dx} = 0$.
- Imports: A response is eligible to earn all 3 points in part (d) with a point of the form $\left(\frac{\pi}{2}, k\right)$ with $k > 0$, imported from part (c).

Alternate Solution for part (d)	Scoring for Alternate Solution	
For the function $y = f(x)$ near the point $\left(\frac{\pi}{2}, 2\right)$, $4y - \sin x > 0$ and $y > 0$.	Considers sign of $4y - \sin x$	**1 point**
Thus, $\frac{dy}{dx} = \frac{y \cos x}{4y - \sin x}$ changes from positive to negative at $x = \frac{\pi}{2}$.	$\frac{dy}{dx}$ changes from positive to negative at $x = \frac{\pi}{2}$	**1 point**
By the First Derivative Test, f has a relative maximum at the point $\left(\frac{\pi}{2}, 2\right)$.	Conclusion	**1 point**

Scoring notes:

- The first point for considering the sign of $4y - \sin x$ may also be earned by stating that $4y - \sin x$ is not equal to zero.
- The second and third points can be earned without the first point.
- To earn the second point a response must state that $\frac{dy}{dx}$ (or $\cos x$) changes from positive to negative at $x = \frac{\pi}{2}$.
- The third point cannot be earned without the second point.
- A response that concludes there is a minimum at this point does not earn the third point.

Total for part (d) 3 points

Total for question 5 9 points

5 5 5 5 5 NO CALCULATOR ALLOWED 5 5 5 5 5

Answer QUESTION 5 parts (a) and (b) on this page.

Response for question 5(a)

$2y^2 - 6 = y \sin x$

$4yy' = y\cos x + y' \sin x$

$4yy' - y' \sin x = y \cos x$

$y'(4y - \sin x) = y \cos x$

$y' = \dfrac{y\cos x}{4y - \sin x}$

$\dfrac{dy}{dx} = \dfrac{y \cos x}{4y - \sin x}$

Response for question 5(b)

b. $\dfrac{dy}{dx}$ At $(0, \sqrt{3}) = \dfrac{\sqrt{3}\cos(0)}{4\sqrt{3} - \sin(0)} = \dfrac{\sqrt{3}(1)}{4\sqrt{3} - (0)} = \dfrac{\sqrt{3}}{4\sqrt{3}} = \dfrac{1}{4}$

$y - \sqrt{3} = \frac{1}{4}(x - 0)$

Answer QUESTION 5 parts (c) and (d) on this page.

Response for question 5(c)

$2y^2 - 6 = y\sin x$

$\frac{dy}{dx} = \frac{y\cos x}{4y - \sin x}$

$y \neq 0$: ~~$2(0)^2$~~ $- 6 = 0 \sin x$

$y \neq 0$

$0 = \frac{y\cos x}{4y - \sin x}$

$x = \frac{\pi}{2}$: $2y^2 - 6 = y\sin(\frac{\pi}{2})$

$2y^2 - 6 = y(1)$

$0 = y\cos x$

$2y^2 - y - 6 = 0$

$(2y+3)(y-2) = 0$

$y \neq 0 \quad x \neq 0$

since $\cos(0) = 1$ and $0 \neq 1$

$0 = y\cos x$

$x = \frac{\pi}{2}$

$y = 2 \qquad 2y + 3 = 0$

$2y = -3$

~~$y = \frac{3}{2}$~~

f(x) has a horizontal tangent at $(\frac{\pi}{2}, 2)$ ~~and $(\frac{\pi}{2}, -\frac{3}{2})$~~

Response for question 5(d)

$\frac{dy}{dx} = \frac{y\cos x}{4y - \sin x}$

Since $\frac{d^2y}{dx^2} < 0$ at $(\frac{\pi}{2}, 2)$, f has a relative maximum at that point.

$\frac{d^2y}{dx^2} = \frac{(4y - \sin x)[y(-\sin x) + y'\cos x] - (y\cos x)(4y' - \cos x)}{(4y - \sin x)^2}$

At $(\frac{\pi}{2}, 2)$: $\frac{d^2y}{dx^2} = \frac{(4 \cdot 2 - 1)[2(-1) + \text{~~0 cos x~~}] - \text{~~(2(0))(4(0) - 0)~~}}{(4(2) - 1)^2}$

$\frac{d^2y}{dx^2} = \frac{(8-1)(-2)}{(8-1)^2} = \frac{7(-2)}{7^2} = -\frac{2}{7}$

↑ concave down

Answer QUESTION 5 parts (a) and (b) on this page.

Response for question 5(a)

$2y^2 - 6 = (y\sin x)$

$4y\frac{dy}{dx} = (y\cos x) + (\sin x \frac{dy}{dx})$

$(4y\frac{dy}{dx} - \sin x\frac{dy}{dx}) = y\cos x$

$\frac{dy}{dx}(4y - \sin x) = y\cos x$

$\frac{dy}{dx} = \frac{y\cos x}{4y - \sin x}$

Response for question 5(b)

$\frac{dy}{dx} = \frac{y\cos x}{4y\sin x}\Big|_{(0,\sqrt{3})} = \frac{3\cos(0)}{4(\sqrt{3}\sin(0))} = \frac{\sqrt{3}}{}$

$y - \sqrt{3} = \frac{\sqrt{3}\cos(0)}{4(\sqrt{3}\sin(0))}(x - 0)$

5 5 5 5 5 **NO CALCULATOR ALLOWED** 5 5 5 5 5

Answer QUESTION 5 parts (c) and (d) on this page.

Response for question 5(c)

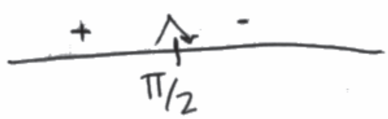
$$\frac{dy}{dx} = \frac{y\cos x}{4y - \sin x} = 0$$

$$y\cos x = 0$$

$$\cos(\pi/2) = 0$$

at the point $(\pi/2, 1)$ the line tangent to the curve is horizontal

Response for question 5(d)



On the point $(\pi/2, 1)$ f has a relative maximum because the values of $f'(x)$ switch from positive to negative at this x-value.

Answer QUESTION 5 parts (a) and (b) on this page.

Response for question 5(a)

$$4y\frac{dy}{dx} = \frac{dy}{dx}\sin x + y\cos x$$

$$4y\frac{dy}{dx} - \sin x\frac{dy}{dx} = y\cos x$$

$$\frac{dy}{dx} = \frac{y\cos x}{4y - \sin x}$$

Response for question 5(b)

$$2(3) - 6 = \sqrt{3}\sin 0$$

$$6 - 6 = 0$$

$$y - \sqrt{3} = \frac{y\cos x}{4y - \sin x}(x)$$

$$y = \frac{y\cos x}{4y - \sin x}(x) + \sqrt{3}$$

5 5 5 5 5 NO CALCULATOR ALLOWED 5 5 5 5 5

Answer QUESTION 5 parts (c) and (d) on this page.

Response for question 5(c)

$2y^2-6=y$

$x=\pi$

$2y^2-6=y\,(1)$

$2y^2-6$

$2(y^2-3)$ $=\pm\sqrt{3}$

$\frac{dy}{dx}=\frac{y\cos\pi}{4y-\sin\pi}=0$

$(0,\sqrt{3})$

$x=\pi$

$(\pi,$

$\frac{dy}{dx}=0$ $\frac{y\cos x}{4y-\sin x}$

Response for question 5(d)

$\frac{d^2y}{dx^2}=\frac{(4y-\sin x)(\cos\pi\frac{dy}{dx})(-y\sin x)-(4\frac{dy}{dx}+\cos x)(y\cos\pi)}{(4y-\sin x)^2}$

if $\frac{d^2y}{dx}>0$

Question 5

Note: Student samples are quoted verbatim and may contain spelling and grammatical errors.

Overview

In this problem $y = f(x)$ is an implicitly defined function whose curve is given by $2y^2 - 6 = y \sin x$ for $y > 0$. In part (a) students were asked to show that $\frac{dy}{dx} = \frac{y \cos x}{4y - \sin x}$, which can be done using implicit differentiation. In part (b) students were asked to write an equation for the tangent line at the point $(0, \sqrt{3})$. A correct response would evaluate the derivative given in part (a) at the point $(0, \sqrt{3})$ and then write the equation of a line through the given point with slope equated to the evaluated derivative.
In part (c) students were asked to find the coordinates of the point where the line tangent to the curve is horizontal for $0 \le x \le \pi$ and $y > 0$. A correct response would set the slope of the tangent line, $\frac{dy}{dx}$, equal to zero, then determine that $y \cos x = 0$ when $x = \frac{\pi}{2}$. The response should then use the given equation $2y^2 - 6 = y \sin x$ to find $y = 2$ when $x = \frac{\pi}{2}$, which results in the point with coordinates $\left(\frac{\pi}{2}, 2\right)$.
In part (d) students were asked to determine and justify whether the function f has a relative minimum, a relative maximum, or neither at the point found in part (c): $\left(\frac{\pi}{2}, 2\right)$. A correct response would use the quotient rule to find $\frac{d^2y}{dx^2}$, determine the sign of $\frac{d^2y}{dx^2}$ at the critical point $\left(\frac{\pi}{2}, 2\right)$, and conclude that f has a relative maximum at this point.

Sample: 5A
Score: 9

The response earned 9 points: 2 points in part (a), 1 point in part (b), 3 points in part (c), and 3 points in part (d). In part (a) the response earned the first point in line 2 with a correct implicit differentiation equation. Having earned the first point, the response is eligible to earn the second point. The response earned the second point with correct algebraic work in lines 3, 4, 5, and 6, verifying the given expression for $\frac{dy}{dx}$. Note that the response would have earned the second point with either line 3 or line 4 leading to either line 5 or line 6. In part (b) the response earned the point for a correct equation of the tangent line on line 2. In part (c) the response earned the first point at the beginning of line 2 for setting the given expression for $\frac{dy}{dx}$ equal to 0. The response would have earned the second point at the beginning of line 6 for the equation $x = \frac{\pi}{2}$ with no other x-values present. In this case, the response earned the second and third points with the commitment to the single ordered pair $\left(\frac{\pi}{2}, 2\right)$ in the circled statement. In part (d) the response earned the first point in line 2 for an attempt to find $\frac{d^2y}{dx^2}$ using the quotient rule. The response earned the second point for a correct expression for $\frac{d^2y}{dx^2}$ found on line 2 followed by a correct evaluation

Question 5 (continued)

of $\frac{d^2y}{dx^2}$ at the point $\left(\frac{\pi}{2}, 2\right)$ in line 3 with no subsequent errors. The response earned the third point with the circled statement, presenting a correct conclusion with the justification "$\frac{d^2y}{dx^2} < 0$ at $\left(\frac{\pi}{2}, 2\right)$."

Sample: 5B
Score: 6

The response earned 6 points: 2 points in part (a), no points in part (b), 2 points in part (c), and 2 points in part (d). In part (a) the response earned the first point in line 2 with a correct implicit differentiation equation. Having earned the first point, the response is eligible to earn the second point. The response would have earned the second point with the work in either line 3 or line 4 leading to line 5. In this case, the response earned the second point with correct algebraic verification work in lines 3, 4, and 5. In part (b) the response did not earn the point because there is an error in the presentation of the slope value, missing the subtraction in the denominator of the expression. In part (c) the response earned the first point in line 1 by setting $\frac{dy}{dx}$ equal to $0.$ The response earned the second point in line 4 with the correct x-value of $\frac{\pi}{2}$ presented in the ordered pair. The response presents an incorrect y-value of 1 in the ordered pair and did not earn the third point. In part (d) the response does not present an attempt to find the second derivative as required in the primary solution shown in the scoring guide, so the alternate solution is considered. The response does not reference the sign of $4y - \sin x$ and did not earn the first point. The response is eligible for the second and third points because the response references a point with the correct x-value of $\frac{\pi}{2}.$ The response earned the second and third points with the statement "f has a relative maximum because the values of $f'(x)$ switch from positive to negative at this x-value." Note that the stem of the question states that $y = f(x),$ thus $f'(x)$ is an acceptable alternative notation for $\frac{dy}{dx}.$

Sample: 5C
Score: 4

The response earned 4 points: 2 points in part (a), no points in part (b), 1 point in part (c), and 1 point in part (d). In part (a) the response earned the first point in line 1 for a correct implicit differentiation of the given equation. Having earned the first point, the response is eligible to earn the second point. The response earned the second point with correct algebraic work in lines 2 and 3. This response demonstrates a minimum amount of verification work required to earn the second point. In part (b) the response did not earn the point. The response does not present a correct numerical expression for the slope in the equation of the tangent line. In part (c) the response earned the first point on the last line for the equation $\frac{dy}{dx} = 0.$ The response does not present the correct x-value, so did not earn the second point. The response does not present a y-coordinate and so did not earn the third point. In part (d) the response earned the first point for an attempt at finding $\frac{d^2y}{dx^2}$ using the quotient rule. The attempt contains errors, so the response is not eligible for the second point. The response presents no further work leading to a consistent conclusion, so the response did not earn the third point.

2021

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AP® Calculus AB

Sample Student Responses and Scoring Commentary

Inside:

Free Response Question 6

☑ Scoring Guideline

☑ Student Samples

☑ Scoring Commentary

Part B (AB): Graphing calculator not allowed
Question 6 — 9 points

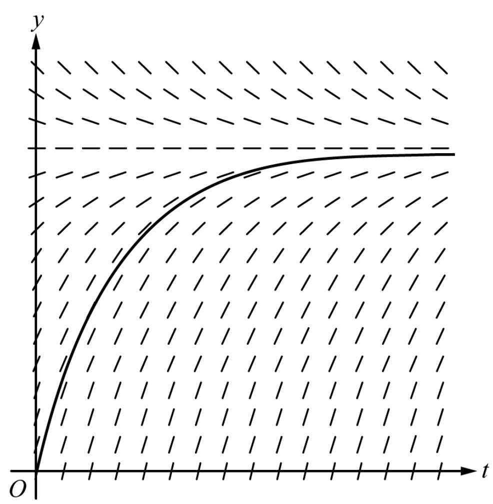
General Scoring Notes
Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

Scoring guidelines and notes contain examples of the most common approaches seen in student responses. These guidelines can be applied to alternate approaches to ensure that these alternate approaches are scored appropriately.

A medication is administered to a patient. The amount, in milligrams, of the medication in the patient at time t hours is modeled by a function $y = A(t)$ that satisfies the differential equation $\frac{dy}{dt} = \frac{12 - y}{3}$. At time $t = 0$ hours, there are 0 milligrams of the medication in the patient.

Model Solution	Scoring

(a) A portion of the slope field for the differential equation $\frac{dy}{dt} = \frac{12 - y}{3}$ is given below. Sketch the solution curve through the point $(0, 0)$.

	Solution curve	**1 point**

Scoring notes:

- To earn the point the solution curve must pass through the point $(0, 0)$, be generally increasing and concave down, and approach the horizontal asymptote from below as t increases. The point is not earned if two or more solution curves are presented.

Total for part (a) 1 point

(b) Using correct units, interpret the statement $\lim_{t\to\infty} A(t) = 12$ in the context of this problem.

Over time the amount of medication in the patient approaches 12 milligrams.	Interpretation	**1 point**

Scoring notes:

- To earn the point the interpretation must include "medication in the patient," "approaches 12," and units (milligrams), or their equivalents.

Total for part (b) 1 point

(c) Use separation of variables to find $y = A(t)$, the particular solution to the differential equation $\frac{dy}{dt} = \frac{12-y}{3}$ with initial condition $A(0) = 0$.

$\frac{dy}{dt} = \frac{12-y}{3} \Rightarrow \frac{dy}{12-y} = \frac{dt}{3}$	Separation of variables	**1 point**
$\int \frac{dy}{12-y} = \int \frac{dt}{3} \Rightarrow -\ln\lvert 12-y\rvert = \frac{t}{3} + C$	Antiderivatives	**1 point**
$\ln\lvert 12-y\rvert = -\frac{t}{3} - C \Rightarrow \lvert 12-y\rvert = e^{-t/3-C}$ $\Rightarrow y = 12 + Ke^{-t/3}$ $0 = 12 + K \Rightarrow K = -12$	Constant of integration and uses initial condition	**1 point**
$y = A(t) = 12 - 12e^{-t/3}$	Solves for y	**1 point**

Scoring notes:

- A response of $\frac{dy}{12-y} = 3\,dt$ is a bad separation and does not earn the first point. However, this response is eligible for the second and third points. It cannot earn the fourth point.
- Absolute value bars are not required in this part.
- A response that correctly separates to $\frac{3\,dy}{12-y} = dt$ but then incorrectly simplifies to $\frac{dy}{4-y} = dt$ earns the first point (for the initial correct separation), is eligible for the second point (for $-\ln\lvert 4-y\rvert = t$, with or without $+C$), but is not eligible for the third or fourth points.
- $+\ln\lvert 12-y\rvert = \frac{t}{3}$ (with or without $+C$) does not earn the second point and is not eligible for the fourth point; $+\ln\lvert 12-y\rvert = \frac{t}{3} + C$ is eligible for the third point.
- In all other cases, the points are earned consecutively—the second point cannot be earned without the first, the third without the second, etc.

Total for part (c) 4 points

(d) A different procedure is used to administer the medication to a second patient. The amount, in milligrams, of the medication in the second patient at time t hours is modeled by a function $y = B(t)$ that satisfies the differential equation $\frac{dy}{dt} = 3 - \frac{y}{t+2}$. At time $t = 1$ hour, there are 2.5 milligrams of the medication in the second patient. Is the rate of change of the amount of medication in the second patient increasing or decreasing at time $t = 1$? Give a reason for your answer.

$\frac{dy}{dt} = 3 - \frac{y}{t+2} \Rightarrow \frac{d^2y}{dt^2} = (-1)\frac{\frac{dy}{dt}(t+2) - y}{(t+2)^2}$	Quotient rule	**1 point**
$B'(1) = 3 - \frac{B(1)}{3} = 3 - \frac{2.5}{3} = \frac{6.5}{3}$ $B''(1) = -\frac{B'(1)\cdot 3 - B(1)}{3^2} = -\frac{6.5 - 2.5}{9} = -\frac{4}{9} < 0$	$B''(1) < 0$	**1 point**
The rate of change of the amount of medication is decreasing at time $t = 1$ because $B''(1) < 0$ and $\frac{d^2y}{dt^2}$ is continuous in an interval containing $t = 1$.	Answer with reason	**1 point**

Scoring notes:

- The first point is for correctly applying the quotient rule to $\frac{y}{t+2}$ or applying the product rule to $y(t+2)^{-1}$. Errors in differentiating the constant, 3, or handling the sign of the second term of $\frac{dy}{dt}$ will result in not earning the second point.
- The second point cannot be earned unless the second derivative $\frac{d^2y}{dt^2}$ is correct.
- For the second point it is sufficient to state the sign of $B''(1)$ is negative with supporting work. If a value is declared for $B''(1)$, it must be correct in order to earn the second point.
- Eligibility for the third point: An attempt at using the quotient rule (or product rule) to find $B''(1)$. In this case the third point will be earned for a consistent conclusion based on the declared value (or sign) of $B''(1)$.

Total for part (d) 3 points

Total for question 6 9 points

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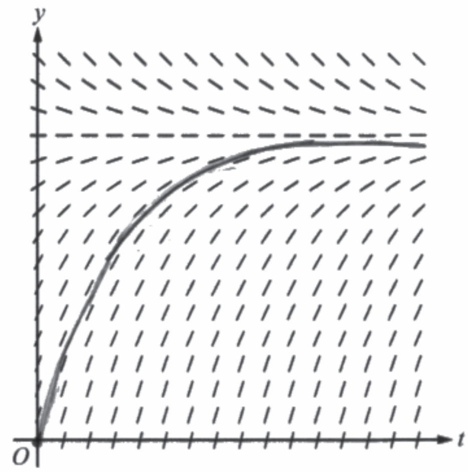
Answer QUESTION 6 parts (a) and (b) on this page.

Response for question 6(a)



Response for question 6(b)

As time goes to infinity, the amount of medication in the patient, in milligrams, is approaching 12.

6 6 6 6 6 NO CALCULATOR ALLOWED 6 6 6 6 6

Answer QUESTION 6 parts (c) and (d) on this page.

Response for question 6(c)

$\frac{dy}{dt} = \frac{12-y}{3}$

$\int \frac{dy}{12-y} = \int \frac{dt}{3}$

$-\ln|12-y| = \frac{1}{3}t + C$

$-\ln|12-0| = \frac{1}{3}(0) + C$

$C = -\ln 12$

$-\ln|12-y| = \frac{1}{3}t - \ln 12$

$\ln|12-y| = -\frac{1}{3}t + \ln 12$

$e^{\ln|12-y|} = e^{-\frac{1}{3}t + \ln 12}$

$12-y = e^{-\frac{1}{3}t} \cdot e^{\ln 12}$

$12-y = 12e^{-\frac{1}{3}t}$

$-y = 12e^{-\frac{1}{3}t} - 12$

$y = -12e^{-\frac{1}{3}t} + 12$

Response for question 6(d)

~~$\frac{dy}{dt}\Big|_{(t,y)=(1,2.5)} = 3 - \frac{2.5}{1+2} = 3 - \frac{2.5}{3} = 3 - \frac{5}{6} = +\#$~~

$\frac{dy}{dt} = 3 - \frac{y}{t+2}$

$\frac{d^2y}{dt^2} = -\frac{dy/dx\,(t+2) - 1(y)}{(t+2)^2}$

$\frac{d^2y}{dt^2} = -\frac{(3-\frac{y}{t+2})(t+2) - y}{(t+2)^2}$

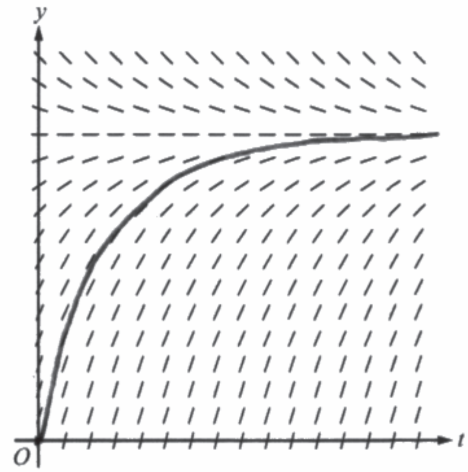
$\frac{d^2y}{dt^2}\Big|_{(1,2.5)} = -\frac{(3-\frac{2.5}{1+2})(1+2) - 2.5}{(1+2)^2} = -\frac{(3-\frac{5}{6})(3) - 2.5}{3^2} = -\#$

The rate of change of the amount of medication in the second patient is decreasing at $t=1$ because $\frac{d^2y}{dt^2}\Big|_{(t,y)=(1,2.5)}$ is < 0.

6 6 6 6 6 NO CALCULATOR ALLOWED 6 6 6 6 6

Answer QUESTION 6 parts (a) and (b) on this page.

Response for question 6(a)



Response for question 6(b)

$\lim_{t \to \infty} A(t) = 12$ means that 12 milligrams is the maximum amount of medication a patient can receive.

Answer QUESTION 6 parts (c) and (d) on this page.

Response for question 6(c)

$3dy = (12-y)dt$

$\int \frac{3}{12-y}dy = \int dt$

$-3\ln|12-y| = t + c$

$e^{\ln|12-y|} = e^{-\frac{1}{3}t + \frac{1}{3}c}$

$12 - y = e^{-\frac{1}{3}t - \frac{1}{3}c}$

$y = 12 - e^{-\frac{1}{3}t - \frac{1}{3}c}$

$0 = 12 - e^{-\frac{1}{3}(0) - \frac{1}{3}c}$

$0 = 12 - e^{-\frac{1}{3}c}$

$e^{-\frac{c}{3}} = 12$

$-\frac{c}{3} = \ln 12$

$c = -3\ln 12$

$y = 12 - e^{-\frac{t}{3} + \ln 12}$

Response for question 6(d)

$\frac{d^2y}{dt^2} = -\frac{(t+2) - y}{(t+2)^2}$

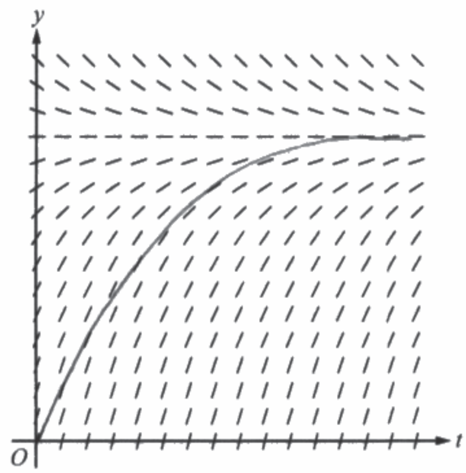
$\frac{d^2y}{dt^2} = -\frac{(1+2) - 2.5}{(1+2)^2} = -\frac{0.5}{9} = -\frac{1}{18}$

The rate of change of the amount of medication is decreasing because the second derivative is less than 0 at $t=1$.

6 6 6 6 6 NO CALCULATOR ALLOWED 6 6 6 6 6

Answer QUESTION 6 parts (a) and (b) on this page.

Response for question 6(a)



Response for question 6(b)

As t aproaches ∞ the amount of medication in the patient aproaches 12 milligrams of medication.

6 6 6 6 6 NO CALCULATOR ALLOWED 6 6 6 6 6

Answer QUESTION 6 parts (c) and (d) on this page.

Response for question 6(c)

$$\frac{dy}{dt} = \frac{12-y}{3}$$

$$\frac{dy}{dt} + \frac{y}{3} = 4$$

$$A(t) + \frac{y^2}{6} = 4y$$

$$\boxed{A(t) = 4y - \frac{y^2}{6}}$$

Response for question 6(d)

$$\frac{dy}{dt} = 3 - \frac{y}{t+2}$$

$$\frac{dy}{dt} \text{ at } t=1 = 3 - \frac{2.5}{3}$$

$$\frac{d^2y}{dt^2} = -\left(\frac{(t+2)\left(\frac{dy}{dx}\right) - (y)(1)}{(t+2)^2}\right)$$

$$= -\left(\frac{(3)\left(3-\frac{2.5}{3}\right) - (2.5)}{9}\right)$$

The rate of change of the amount of medication in the second patient is decreasing, as $B''(t)$ is negative.

Question 6

Note: Student samples are quoted verbatim and may contain spelling and grammatical errors.

Overview

In this problem a function $y = A(t)$ models the amount of medication, in milligrams, in a patient at time t hours. This function satisfies the differential equation $\frac{dy}{dt} = \frac{12 - y}{3}$, and at time $t = 0$ hours, there are 0 milligrams of medication in the patient.

In part (a) students were shown a portion of the slope field for the given differential equation and asked to sketch the solution curve through the point $(0, 0)$. A correct response would draw a single increasing, concave down curve starting at $(0, 0)$, approaching the horizontal asymptote with slopes equal to zero from below.

In part (b) students were asked to interpret the statement $\lim_{x\to\infty} A(t) = 12$ using correct units in this context. A correct response would indicate this statement means that over time the amount of medication in the patient approaches 12 milligrams.

In part (c) students were asked to use separation of variables to find the particular solution $y = A(t)$ with $A(0) = 0$. A correct response should separate the variables, integrate, and use the initial condition $A(0) = 0$ to resolve the constant of integration and arrive at the solution $A(t) = 12 - 12e^{-t/3}$.

In part (d) a second function $y = B(t)$, which satisfies $\frac{dy}{dt} = 3 - \frac{y}{t+2}$, is introduced as a model for the amount of medication in a second patient at time t hours. At time $t = 1$ hour, there are 2.5 milligrams of medication in the second patient. Students were asked whether the amount of medication in the patient is increasing or decreasing at time $t = 1$. A correct response would use the quotient rule to compute $B''(t) = \frac{d^2y}{dt^2}$, determine that $B'(1) \neq 0$ and $B''(1) < 0$, and then would conclude the amount of medication is decreasing.

Sample: 6A
Score: 9

The response earned 9 points: 1 point in part (a), 1 point in part (b), 4 points in part (c), and 3 points in part (d). In part (a) the response earned the point because the solution curve passes through $(0, 0)$, is increasing and concave down, and approaches the horizontal asymptote. In part (b) the response was earned because it references both "medication in the patient in milligrams" and "approaching 12." In part (c) the response earned the first point on line 2 on the left side for a correct separation of variables. The second point was earned on line 3 on the left side for correct antiderivatives. The third point was earned for the "$+ c$" on line 3 on the left side with the use of the initial condition on line 4 on the left side. The fourth point was earned for a correct solution presented in the box on the last line on the right side. In part (d) the response earned the first point on line 2 for the correct second derivative expression. It is unclear on lines 3 and the first part of line 4 whether or not the leading negative sign is in the numerator or in front of the fraction; however, it is clear on the final presented numerical value. The second point was earned for the correct numeric expression of the second derivative. The expression does not have to be simplified to earn the point. The third point was earned for the answer decreasing with the reasoning based on the sign of the second derivative at $(1, 2.5)$.

Question 6 (continued)

Sample: 6B
Score: 6

The response earned 6 points: 1 point in part (a), no points in part (b), 4 points in part (c), and 1 point in part (d). In part (a) the response earned the point because the solution curve passes through $(0, 0)$, is increasing and concave down, and approaches the horizontal asymptote. In part (b) the point was not earned because the response states "12 milligrams is the maximum amount" instead of "approaches 12 milligrams" and the response refers to the amount of medication "a patient can receive" instead of the amount of medication in the patient. In part (c) the response earned the first point on line 2 on the left side for the correct separation of variables. The second point was earned on line 3 on the left side for the correct antiderivatives. The third point was earned for the "$+ c$" on line 3 on the right side with the use of the initial condition on line 2 on the right side. The fourth point was earned for a correct form of the solution presented in the box on the last line. In part (d) the first point was not earned because the expression for the second derivative is incorrect. There is a missing $\frac{dy}{dt}$ in the numerator. The response is not eligible for the second point because the second point cannot be earned without the correct second derivative. The third point was earned for the conclusion, decreasing, based on the reasoning "because the second derivative is less than 0 at $t = 1$."

Sample: 6C
Score: 4

The response earned 4 points: 1 point in part (a), 1 point in part (b), no points in part (c), and 2 points in part (d). In part (a) the response earned the point because the solution curve passes through $(0, 0)$, is increasing and concave down, and approaches the horizontal asymptote. In part (b) the response earned the point because it mentions "amount of medication in the patient" and "aproaches 12 milligrams." In part (c) the response did not earn the first point because it does not separate the variables. A response that does not separate the variables is not eligible for any other points in part (c). In part (d) the response earned the first point on line 2 for the correct second derivative expression. The second point was not earned because equating the symbolic second derivative to the numeric second derivative expression causes a linkage error. In this case, if the linkage error had not occurred, the point would be earned for the correct unsimplified value of the second derivative at $t = 1$. The third point was earned for the answer decreasing with the reason "$B''(t)$ is negative."